ESSAYS IN MECHANISM DESIGN

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A DISSERTATION

in

Applied Economics

For the Graduate Group in Managerial Science and Applied Economics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2025

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ACKNOWLEDGEMENT

Thank you to my advisors, Eduardo Azevedo, Alex Rees-Jones, and Juuso Toikka for their patient and insightful guidance.

ABSTRACT

ESSAYS IN MECHANISM DESIGN

Ilan Wolff

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This dissertation is comprised of 3 papers. Here are the abstracts for the 3 papers.

Welfare Properties of the Taiwan Mechanism: This paper analyzes the welfare properties of the mechanism Taiwan uses to match students to high schools. I study the Taiwan mechanism under a theoretical model where students share the same ordinal rankings, and a single lottery determines schools' preferences. In this setting, all Taiwan mechanisms ex ante weakly Pareto dominate deferred acceptance for strategizing students. Additionally, in my model, there always exists a Taiwan mechanism that provides greater welfare than deferred acceptance to students who do not strategize. Using Cambridge Public School data, I demonstrate that the Taiwan mechanism outperforms deferred acceptance for a range of deduction rules.

Broad Validity of the First-Order Approach in Moral Hazard: The first-order approach (FOA) is the main tool in the study of the pure moral hazard principal-agent problem. Although many existing results rely on the FOA, its validity has been established only under relatively restrictive assumptions. We contribute three main findings.

First, we demonstrate in a broad array of examples that the FOA frequently fails when the agent's reservation utility is low (such as in principal-optimal contracts). However, the FOA holds when the agent's reservation utility is at least moderately high (such as in competitive settings where agents receive high rents).

Second, our main theorem formally shows that the FOA is valid in a standard limited liability

model when the agent's reservation utility is sufficiently high. The theorem also establishes existence and uniqueness of the optimal contract.

Third, we use the FOA to derive tractable optimal contracts across a broad array of settings. These contracts are both simple and intuitive, and under log utility, they are piecewise linear for numerous common output distributions (including Gaussian, exponential, binomial, Gamma, and Laplace).

Heterogeneous value-added for priority design in New York City high schools: School districts that use centralized matching markets can design the schools' admissions criteria to prioritize students who benefit the most from attending the school. I study heterogeneous value-added in New York City high schools to determine which students have high value-added from attending which types of schools. Students with high middle school standardized test scores benefit the most from attending academically competitive high schools. Weaker students, as well as students with disabilities and English language learners, perform better in less competitive high schools. My results suggest that NYC could increase aggregate value-added by adopting a less coarse academic priority structure at the top high schools, so that more students with high heterogeneous value-added can attend those schools.

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CHAPTER 1

WELFARE PROPERTIES OF THE TAIWAN MECHANISM

1.1. Introduction

Trade-offs between equity and efficiency are central to the literature on school choice matching market design. Deferred acceptance's strategic simplicity promotes equity by preventing "sophisticated" students who strategize optimally from exploiting "sincere" students who submit their true preferences (Abdulkadiroglu and Sonmez, 2003; Abdulkadiroglu et al., 2006; Pathak and Sonmez, 2008). However, more strategically complex mechanisms, like the Boston mechanism, may produce matchings with greater aggregate welfare than deferred acceptance (Abdulkadiroglu et al., 2011b; Featherstone and Niederle, 2008; Miralles, 2008). This paper studies the welfare properties of the Taiwan mechanism, a lesser-known mechanism first analyzed by Dur et al. (2022), which Taiwan uses to match students to high schools. I show that a carefully selected Taiwan mechanism, rather than deferred acceptance, maximizes the welfare of sincere students, prompting a reconsideration of deferred acceptance's status as the starting point for negotiations between equity and efficiency.

The Taiwan mechanism affords school districts the flexibility to balance equity and efficiency by employing a continuous deduction rule, parameterized by a vector, λ, which determines the extent to which ranking a school higher gives a student priority at that school. When $\lambda = 0$, students receive no special treatment for ranking a school highly, just like in deferred acceptance. When $\lambda_j - \lambda_i$ is sufficiently large for all $i < j$, students who rank a school higher are always admitted before students who rank the school lower, just like in the Boston mechanism. The deduction rule also allows school districts to compromise continuously between equity and efficiency. A small deduction rule punishes strategic errors less, thereby maintaining fairness between sincere and sophisticated students. However, larger deduction rules provide students with stronger incentives to omit marginally preferred competitive schools from their rank order lists, increasing the mechanism's efficiency.

School districts may think they must choose between the Boston mechanism's efficiency and deferred acceptance's equity, and that deferred acceptance is best for sincere students (Payzant, 2005).

Much of the literature agrees that the Boston mechanism puts sophisticated students at an advantage. For example, Pathak and Sonmez (2008, p.1646) show "that the Boston mechanism favors sophisticated parents at Pareto-dominant Nash equilibrium" and He (2017), Basteck and Mantovani (2018), Dur et al. (2018), and König et al. (2019) reach similar conclusions. Similarly, one might think that sincere students prefer deferred acceptance ($\lambda = 0$) to Taiwan mechanisms with $\lambda > 0$ because larger deduction rules more harshly punish strategic errors. To the contrary, I show that, under reasonable assumptions, the efficiency gains for sincere students from marginally increasing λ above 0 outweigh the costs. School districts can, therefore, always choose a deduction rule so that the Taiwan mechanism provides sincere students with greater aggregate welfare than deferred acceptance. If school districts must choose between the Boston mechanism and deferred acceptance, then either option may be justified by the school district's normative judgments about trade-offs between efficiency and equity. However, if the Taiwan mechanism is added to the school district's choice set, and the model's assumptions hold, then there is always a Taiwan mechanism that provides greater welfare for sincere students than deferred acceptance. Concerns about the distribution of welfare between sincere and sophisticated students, therefore, cannot rationalize deferred acceptance (districts may still prefer deferred acceptance for other reasons which I discuss below).

The theoretical results only show that there exists a deduction rule such that the Taiwan mechanism outperforms deferred acceptance for sincere students. If the set of deduction rules that sincere students prefer is small and depends on unknown parameters, mechanism designers may fear that they will inadvertently select a deduction rule that harms sincere students. To address this concern, I use Cambridge Public School data to demonstrate empirically that the Taiwan mechanism robustly outperforms deferred acceptance for sincere students with a wide range of λ. I estimate students' preferences using the methodology outlined in Agarwal and Somaini (2018), compute the equilibrium, and estimate sincere and sophisticated students' welfare. Sophisticated students' welfare gains are substantially larger, but both sincere and sophisticated students attain higher utility under the Taiwan mechanism with a wide range of λ than under deferred acceptance.

Many papers advocate for deferred acceptance by showing that, in the simple two-type model,

sincere students prefer deferred acceptance to the Boston mechanism. I demonstrate that if our goal is to maximize sincere students' welfare, and the model's assumption hold, then we should never choose deferred acceptance. Of course, there are other reasons a school system might prefer deferred acceptance. The school system may value strategy-proofness for its own sake, or want a stable matching, especially if the schools' priorities reflect merit, like exam scores. Alternatively, the school system may believe that under the Taiwan mechanism, some students would neither submit their true preferences nor the strategically optimal rank order list. Indeed, both Kapor et al. (2020) and De Haan et al. (2023) demonstrate that, under the Boston mechanism, some students do not submit their true preferences and make costly strategic errors. A desire to protect these students may provide a much stronger justification for deferred acceptance, but without a better understanding of why and how these students err, comparing welfare between strategy-proof and manipulable mechanisms is difficult. Modeling these biases and studying how these biases impact the distribution of welfare in manipulable mechanisms like the Taiwan mechanism is an important avenue for future research.

1.2. Model and Mechanisms

In this section, I present the model and elucidate the three mechanisms for solving school choice matching problems that I discuss in this paper: the Taiwan mechanism, deferred acceptance and the Boston mechanism. The discussion focuses on the Taiwan mechanism both because it is the least well known of the mechanisms and because deferred acceptance and the Boston mechanism are special cases of the Taiwan mechanism.

I follow Abdulkadiroglu et al. (2015) and Azevedo and Leshno (2016) by considering matching problems with a finite number of schools and a measure 1 continuum of students. The set of schools is $S = \{s_1, \ldots, s_N\}$ and I denotes the continuum of students. Each school, s, has capacity q_s. The sum of the capacities of schools 1 through $N-1$ is less than 1. The final school is an outside option available to all students and it has capacity 1. Every student has a cardinal utility vector, $v = (v_1, \ldots, v_N)$, governed by the joint CDF F, and a corresponding continuous PDF, f, which is bounded above. The students have the same ordinal preferences for the schools, but heterogeneous

cardinal preferences: for every student, $v_1 = 1 \geq v_2 \geq \ldots \geq v_N = 0$. A real number, l_i, represents the schools' common preference for Student i, so that $i \succsim_s j$ if and only if $l_i > l_j$. The collection of the school's preferences is denoted $L = (l^1, \ldots, l^N)$. This notation may seem redundant because the schools' preferences are common, but it is necessary to define the deferred acceptance algorithm, which I will apply to heterogeneous preferences. The schools' preferences are unknown to the students and distributed $U[0, 1]$. A natural interpretation is that the schools are indifferent among the students, and break ties with a shared lottery. I refer to the schools' preferences as lottery numbers, in accordance with this interpretation.

Students and schools submit preferences to a mechanism which matches the students to schools. Schools must submit their true preferences whereas students may submit any monotonically increasing rank order list that includes the outside option, N. Restricting the students' choice set simplifies the problem and is without loss because students always have a best response in this set (Lemma A.1). When convenient, I write the rank order list without including the outside option because it must be the final school on all rank order lists. For example, if there are 5 schools, I may write $(1, 3)$ to represent the rank order list $(1, 3, 5)$. The set of such rank order lists is $\mathcal{R}$, the collection of students' submitted rank order lists is R and Student i's submitted rank order list is r_i. The mechanism is a function which takes in (R, L, q) and returns a matching, $\mu : I \to S$ such that $|\mu^{-1}(s)| \leq q_s \ \forall s \in S$. In words, the mechanism matches each student to a school, and the mechanism respects the schools' capacity constraints.

I begin my discussion of the mechanisms by explaining the deferred acceptance algorithm, first discovered by Gale and Shapley (1962), which is closely related to all three mechanisms considered in this paper.

Deferred Acceptance Algorithm:

Step 1:

- Each student, i, applies to their top-choice school according to r_i.

- Each school, s, tentatively accepts its most preferred applicants according to l^s up to the school's capacity constraint, q_s.

Step K:

- If Student i is not currently tentatively accepted, Student i applies to their most preferred school at which they have not yet been rejected, according to r_i.
- If Student i is tentatively accepted by School s, Student i reapplies to School s.
- Each school, s, tentatively accepts its most preferred applicants up to its capacity constraint, q_s.

The algorithm terminates when there are no more rejections. Students are assigned to the school they applied to in the final step of the algorithm.

The Taiwan mechanism applies the deferred acceptance algorithm to the schools' preferences after using a deduction rule to adjust the schools' preferences. The deduction rule, parameterized by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, where $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_N$, gives priority to students who rank the school highly by subtracting λ_k from a school's preference for students who rank it kth. Formally, School s's final cardinal evaluation of student i is $\hat{l}_i^s = l_i - \lambda_{r_i(s)}$, where $r_{i(s)}$ is Student i's ranking of School s. For example, if Student i ranks School s 3rd, then $\hat{l}_i^s = l_i - \lambda_3$. The Taiwan mechanism applies the deferred acceptance algorithm to the schools' final cardinal rankings, $\hat{L}$, and the students' submitted preferences, R.

The Taiwan mechanism increases the probability that students who rank a school highly are admitted because $\lambda_1 \leq \ldots \leq \lambda_N$. If Student j applies in an earlier round, Student j may be admitted to School s over Student i even if Student i has the higher lottery number because $l_j - \lambda_{r_j(s)}$ may be greater than $l_i - \lambda_{r_i(s)}$. The Taiwan mechanism is not strategy-proof; if a student thinks they are unlikely to be admitted to their top-choice school, they may rank a different school first in order to increase their probability of admission to the other school.

As Dur et al. (2022) show, both deferred acceptance and the Boston mechanism are particular instances of the Taiwan mechanism. Under deferred acceptance, students who apply to School s in earlier rounds are given no advantage over students who apply to School s in later rounds. This is simply the Taiwan mechanism with $\lambda = (0, \ldots, 0)$. Under the Boston mechanism, students who apply in earlier rounds are always admitted ahead of students who apply in later rounds. The Taiwan mechanism with $\lambda = (0, d, \ldots, (N-1)d)$ for $d > max_i(l_i) - min_i(l_i)$ implements the Boston mechanism. The restriction on d ensures that d is sufficiently large such that even the student with the lowest lottery number will be admitted over the student with the highest lottery number, if the student with the lowest lottery number applies in an earlier round.

Some students may erroneously believe that the Taiwan mechanism is strategy-proof. I assume there are two kinds of students: sophisticated students and sincere students. Sophisticated students understand that the mechanism is not strategy-proof, so they choose r_i to maximize their expected utility. Sincere students, on the other hand, rank the schools according to their true preferences. The fraction of sophisticated students is p_{soph} and the fraction of sincere students is $1 - p_{soph}$.

The solution concept for this game is a Nash equilibrium. In order to define the equilibrium, I first need to specify how the probabilities of admission at different schools depend on students' actions, which requires some additional notation. The choice density vector, $c = (c^{r_1}, c^{r_2}, \ldots)$, represents the measure of students who submit each rank order list r. Each student can only submit one rank order list, so c is in the unit simplex, which I denote $\mathcal{C}$. The vector of probabilities of admission for students who submit rank order list r when the deduction rule is λ and the choice density vector is c is denoted $P^r(c, \lambda) = (p_1^r(c, \lambda), \ldots, p_N^r(c, \lambda))$. Note that P^r is a function of c and that computing P^r does not require any information about which particular students submit each rank order list. The reason is that the schools treat all students equally, so the identity of the students submitting each rank order list is irrelevant to P^r. In equilibrium, c induces probabilities of admission that themselves induce c^{r_1} students to submit r_1, c^{r_2} students to submit r_2, etc. I now formally define a strategy, the associated choice density vector, and an equilibrium.

Definition 1.1. *A **strategy** is a function, σ, which maps a student's utility, v, to a rank order list,*

r.

Definition 1.2. *The vector, $c(\sigma)$, is the* ***choice density vector induced by the strategy $\boldsymbol{\sigma}$*** *if*

$$\forall r : c^r(\sigma) = (1 - p_{soph})\mathbb{I}(r = (1, \ldots, N)) + p_{soph} \int_{\sigma^{-1}(r)} f(v)\, dv.$$

Definition 1.3. *An* ***equilibrium*** *is a strategy, σ, that best responds to the choice density vector it induces, $c(\sigma)$, which is the case if*

$$\forall v : v \cdot (P^{\sigma(v)}(c(\sigma), \lambda) - P^{r'}(c(\sigma), \lambda)) \geq 0 \;\; \forall r' \in \mathcal{R}.$$

1.3. Motivating Examples

This section presents two simple illustrative examples. The first example has a discrete number of students and demonstrates how increasing the deduction rule may improve efficiency. The second example has a continuum of students and shows how equilibrium strategy and probabilities of admission depend on the deduction rule.

1.3.1. Discrete Example

The following example, which Abdulkadiroglu et al. (2011b) use to demonstrate how the Boston mechanism can improve welfare compared to deferred acceptance, also illustrates how welfare depends on the deduction rule in the Taiwan mechanism. In this example, there are three students and three schools, each with one slot. The students' cardinal preferences are in Table 1.1. The number in column i and row j in Table 1.1 is Student i's utility from attending School j. For example, Student 3's utility at School 2 is 4. The schools share the same cardinal preferences for the students; these preferences are unknown to the students and distributed $U[0, 1]$. As usual, I refer to the schools' cardinal preferences as lottery numbers. For simplicity's sake, this example deviates from the model by assuming a discrete number of students rather than a continuum. This simplification does not change the intuition for how welfare depends on the deduction rule, which the example aims to convey.

Table 1.1: Students' utilities (v_s^i)

	i_1	i_2	i_3
s_1	8	8	6
s_2	2	2	4
s_3	0	0	0

I construct an equilibrium for this example by considering the students' optimal strategies as a function of the deduction rule parameter, $\lambda = (0, \lambda_2, \lambda_3)$. The deduction at the third ranked school, λ_3, is irrelevant because there is no competition for admission to School 3, so I only need to consider how λ_2 affects students' strategies. When $\lambda_2 = 0$, there is no advantage to ranking School 2 first, so all students submit the ranking: $(1, 2, 3)$ and have an equal probability of attending each school. Each student's ex ante expected utility is $3\frac{1}{3}$. As λ_2 increases, the probability of receiving an offer of admission at School 2 conditional on ranking School 2 first increases; if this probability is sufficiently large, Student 3 may submit the rank order list $r = (2)$. Let p_j^r denote the probability of admission to s_j conditional on submitting ranking r, and v_s denote Student 3's utility at School i. Then Student 3 is willing to rank School 2 first if and only if:

$$p_1^{(2)} \cdot v_1 + p_2^{(2)} \cdot v_2 \geq p_1^{(1,2)} \cdot v_1 + p_2^{(1,2)} \cdot v_2. \tag{1.1}$$

Note that $p_1^{(1,2)} = p_2^{(1,2)} = \frac{1}{3}$ because the other students rank School 1 first and School 2 second, and $p_1^{(2)} = 0$ because School 1 is omitted from the rank order list. I assume the rank order list $(2, 1)$ is not in the choice set because it is not monotonically increasing but it is worth observing that this rank order list does not improve the student's chances at School 1. Conditional on ranking School 1 second, Student 3 cannot be accepted at School 1; Student 3 would first need to be rejected from School 2, which requires that Student 3's lottery number is lower than the minimum of Student 1's and Student 2's lottery numbers. This reasoning is the intuition for Lemma A.1 which states that a monotonically increasing rank order list is always a best response.

Plugging in for the probabilities and Student 3's utility and solving for the set of $p_2^{(2)}$ that cause Student 3 to rank School 2 first yields $p_2^{(2)} \geq \frac{5}{6}$. Next, I solve for the values of λ_2 that induce

a sufficiently high $p_2^{(2)}$. Students 1 and 2 both rank School 1 first, and only the rejected student competes with Student 3 for a slot at School 2. The rejected student's adjusted lottery number is the minimum of two uniform lottery numbers minus the deduction rule, λ_2. The CDF of this random variable is $G(x) = 1-(1-x-\lambda_2)^2$. Student 3's probability of admission at School 2 is G evaluated at student 3's adjusted lottery number at school 2, $G(\hat{l}_3^2)$. Student 3's adjusted lottery number at school 2 is $U[0,1]$, so integrating G between $(0,1)$ yields Student 3's probability of admission at School 2. I solve for the values of λ that induce $p_2^{(2)} \geq \frac{5}{6}$:

$$\int_0^1 1-(1-x-\lambda_2)^2 dx \geq \frac{5}{6} \Longrightarrow \frac{2}{3}+\lambda_2-\lambda_2^2 \geq \frac{5}{6}. \tag{1.2}$$

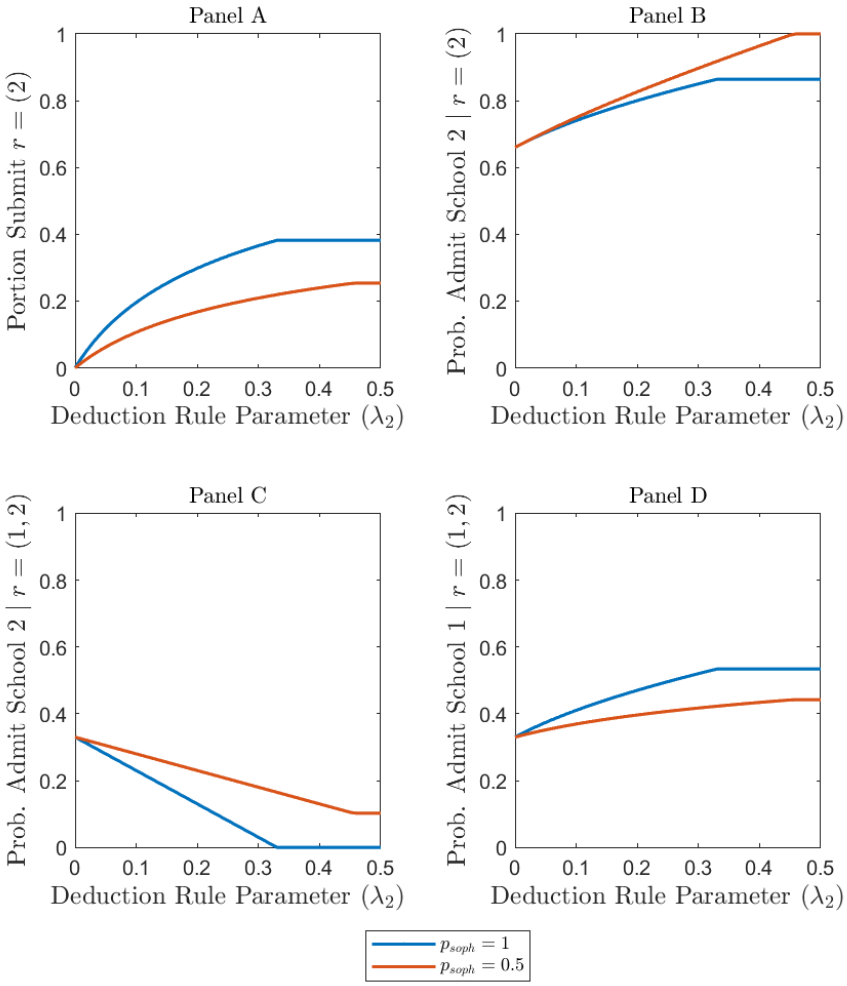
If $\lambda_2 \geq \frac{1}{6}(3+\sqrt{3})$, then it is an equilibrium for Students 1 and 2 to rank School 1 first, and Student 3 to rank School 2 first.[1] It is easy to see that every student's expected welfare increases. The above calculations already show that Student 3 prefers this allocation to the allocation where they have a $\frac{1}{3}$ probability of attending School 1 and a $\frac{1}{3}$ probability of attending School 2. Students 1 and 2 each have a $\frac{1}{2}$ probability of attending School 1, so their expected utility is at least $\frac{1}{2}\cdot 8 = 4$, an improvement over their expected utility of $3\frac{1}{3}$ when $\lambda_2 < \frac{1}{6}(3+\sqrt{3})$.

1.3.2. Continuous Example

Figure 2.5 illustrates how the equilibrium depends on the deduction rule in a simple continuous example. It displays the equilibrium measure of students who submit $r=(2)$ and the probabilities of admission for all $\lambda = (0,\lambda_2), \lambda_2 \in [0,1]$ in a three-school case with $v_2 \sim U[0,1]$, $q_1 = q_2 = \frac{1}{3}$ and $p_{soph} = 1$. The blue line represents a market where all students are sophisticated and the orange lines represents a market where $\frac{1}{2}$ of students are sophisticated. Although the equilibrium is not unique in general (see section A.5 for an example of multiple equilibria), Lemma A.4 demonstrates that any three school case has a unique equilibrium, so the displayed equilibria are unique.

[1]The constructed equilibrium is unique for $\lambda_2 > \frac{1}{6}(3+\sqrt{3})$. To see this, note that Students 1 and 2 prefer $(p_1^{(1,2)}, p_2^{(1,2)}) = (\frac{1}{3}, 0)$ to $(p_1^{(2)}, p_2^{(2)}) = (0,1)$. Regardless of the other students' actions and λ_2, $p_1^{(1,2)} \geq \frac{1}{3}$ and $p_2^{(2)} \leq 1$, so students 1 and 2 always submit $(1,2)$. Conditional on students 1 and 2 ranking school 1 first, Equation (1.1) and Equation (1.2) show that Student 3 strictly prefers to submit (2) at $\lambda_2 > \frac{1}{6}(3+\sqrt{3})$, so the constructed equilibrium is unique. When $\lambda_2 = \frac{1}{6}(3+\sqrt{3})$, Student 3 is indifferent between submitting $(1,2)$ and (2), so there are two pure strategy equilibria and infinitely many mixed strategy equilibria.

Figure 1.1: Illustration of a Simple Equilibrium


Panel A
Portion Submit $r = (2)$
Deduction Rule Parameter (λ_2)
Panel B
Prob. Admit School 2 $\mid r = (2)$
Deduction Rule Parameter (λ_2)
Panel C
Prob. Admit School 2 $\mid r = (1,2)$
Deduction Rule Parameter (λ_2)
Panel D
Prob. Admit School 1 $\mid r = (1,2)$
Deduction Rule Parameter (λ_2)
$p_{soph} = 1$
$p_{soph} = 0.5$

Notes: This figure presents the equilibrium choice density vector and probabilities of admission as a function of the deduction rule for a simple matching problem. Panel A plots the portion of students who rank School 2 first, Panel B plots the probability of admission to School 2 conditional on ranking School 2 first, Panel C plots the probability of admission to School 2 conditional on ranking School 1 first, and Panel D plots the probability of admission to School 1 conditional on ranking School 1 first. Students' preference for School 2, v_2, is distributed uniform [0,1]; $v_1 = 1$ and $v_3 = 0$ for all students. Schools 1 and 2 both have capacity $\frac{1}{3}$. The blue line represents a market where all students are sophisticated and the orange line represents a market where $\frac{1}{2}$ of students are sophisticated.

A few patterns are immediately apparent. Unsurprisingly, $p_2^{(2)}$, the probability of admission to School 2 conditional on ranking School 2 first, is increasing on λ_2. This is because when λ_2 is higher, students who rank School 2 first are given a greater advantage over students who rank School 2 second. As λ_2 increases, $c^{(2)}$ also increases, which means that more students rank School 2 first. The increase in $c^{(2)}$ partially offsets the impact of the increase in λ_2 on $p_2^{(2)}$; students who rank School 2 first have a better chance of admission relative to students who rank School 1 first, but they also must compete with an increasing number of students who rank School 2 first. The probability of admission to School 1 conditional on ranking School 1 first, $p_1^{(1,2)}$, increases on λ_2 because the number of students who rank School 1 first decreases on λ_2, and students who rank School 1 first only compete for admission to School 1 with other students who rank School 1 first. Lastly, $p_2^{(1,2)}$ decreases on λ_2, both because higher values of λ_2 put students who rank School 2 second at a larger disadvantage for admission to School 2 relative to students who rank School 2 first, and because higher values of λ_2 induce more students to rank School 2 first. Once λ_2 is large enough to induce the Boston mechanism equilibrium, all variables become constant functions of λ_2.

1.4. Equilibrium Characterization and Welfare

Theorem 1.1. *An equilibrium exists.*

I prove that an equilibrium exists in section A.1 in the appendix, but briefly explain the intuition here. Existence proofs of equilibria usually identify fixed points in the strategy space and the equilibrium function in Definition 1.3 is a function which maps to a fixed point in the students' strategies. However, finding a fixed point in the students' strategies is difficult because there are infinitely many types of students. Working with the choice density vector, c, is easier because it is finite dimensional. I find a fixed point in the choice density vector, which means that given the probabilities of admission induced by c, it is optimal for c^{r_1} students to submit r_1, c^{r_2} students to submit r_2, etc. I then show that the existence of a fixed point in c implies the existence of an actual equilibrium.

Having established the existence of an equilibrium, I now analyze the equilibrium admission probabilities. I characterize a matrix of ex-ante probabilities of admission, P, as a function of the

choice density vector, c. Each column of P is a vector, P^r, where p_s^r is the probability of admission at School s for students who submit rank order list r. Probabilities of admission at each school follow from the minimum lottery numbers necessary for admission. I use $\bar{L}_s$ to denote the minimum adjusted lottery number necessary for admission to School s. This means that all students who apply to School s in round x and have a lottery number, $l_i > \bar{L}_s + \lambda_x$ will be accepted. The minimum lottery number necessary for admission at s must ensure that s accepts its best q_s applicants, and rejects the remaining $T_s - q_s$, where T_s is School s's total number of applicants, which is equal to the measure of students who are rejected at all schools they rank ahead of s. The CDF of applicants' adjusted lottery numbers, evaluated at $\bar{L}_s$, multiplied by the number of applicants, must equal $T_s - q_s$. I, therefore, first solve for the distribution of students' lottery numbers and then solve for $\bar{L}_s$.

When students apply to s, their lottery numbers are no longer necessarily $U[0,1]$ both because the deduction rule adjusts lottery numbers downwards and because rejection at higher ranked schools reveals information about the students' lottery numbers. The lower bound of the distribution of lottery numbers for a student who ranks s in the $r(s)$th position is $-\lambda_{r(s)}$ because the lower bound of the initial distribution of lottery numbers is 0 and an additional $\lambda_{r(s)}$ is subtracted from the student's lottery number.

Rejections at higher ranked schools restrict the upper bound of the student's lottery numbers. Students who submit r do not apply to s unless all the schools they rank ahead of s reject them. The probability of rejection at all schools ranked ahead of s is $1 - \sum_{t<s} p_t^r$, so rejection at all higher ranked schools implies an unadjusted lottery number less than $1 - \sum_{t<s} p_t^r$. The deduction rule further reduces the upper bound of students' adjusted lottery numbers by $\lambda_r(s)$. The distribution of adjusted lottery numbers for a student who submits rank order list r, and applies to s is, therefore, $U[-\lambda_{r(s)}, 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}]$.

Having derived the distribution of students' lottery numbers, I now derive the cutoff lottery number at each school by setting the number of students the school must reject equal to the CDF of students' lottery numbers evaluated at the cutoff lottery number. The number of students s

rejects, $T_s - q_s$, is equal to the sum across r of the number of students who submit r and apply to s, denoted w_s^r, multiplied by the CDF of a uniform distribution with lower bound $-\lambda_{r(s)}$ and upper bound $1 - \sum_{t<s} p_t^r - \lambda_{r(s)}$ evaluated at $\bar{L}_s$:

$$T_s - q_s = \sum_r w_s^r U[-\lambda_{r(s)}, 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}](\bar{L}_s). \tag{1.3}$$

The measure of students who submit r and apply to s equals the measure of students who submit r and are rejected from all schools they rank ahead of s: $w_s^r = c^r(1 - \sum_{t<s} p_t^r)$. I now plug in for w_s^r, T_s, and the CDF of the uniform distribution to solve for $\bar{L}_s$. The uniform distribution is piece-wise, so plugging in requires separating rank order lists into 3 groups, $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ depending on whether $\bar{L}_s$ is within the distribution of the possible adjusted lottery numbers at s that r induces, less than the lower bound of the distribution that r induces, or greater than the upper bound. I rewrite Equation 1.3 as

$$\sum_{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3} c^r(1 - \sum_{t<s} p_t^r) - q_s = \sum_{\mathcal{R}_1} c^r(1 - \sum_{t<s} p_t^r) + \sum_{\mathcal{R}_2} c^r(\bar{L}_s + \lambda_{r(s)}), \tag{1.4}$$

where

$$\begin{cases} r \in \mathcal{R}_1 & \iff 1 - \sum_{t<s} p_t^r - \lambda_{r(s)} < \bar{L}_s, \\ r \in \mathcal{R}_2 & \iff -\lambda_{r(s)} \leq \bar{L}_s \leq 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}, \\ r \in \mathcal{R}_3 & \iff \bar{L}_s < -\lambda_{r(s)}. \end{cases}$$

Intuitively, $\mathcal{R}_1$ is the set of rank order lists that guarantee rejection from s, $\mathcal{R}_2$, is the set of rank of order lists that induce a probability of admission to s that is between 0 and 1, and $\mathcal{R}_3$ is the set of rank order lists that guarantee acceptance at s.

Simplifying Equation (1.4) yields the following theorem:

Theorem 1.2. *Given any choice density vector, c, the vector of cutoff lottery numbers, $\bar{L}$, satisfies*

$$\bar{L}_s(c) = \frac{\sum_{\mathcal{R}_2,\mathcal{R}_3} c^r(1 - \sum_{t<s} p_t^r) - q_s - \sum_{\mathcal{R}_2} c^r \lambda_{r(s)}}{\sum_{\mathcal{R}_2} c^r}, \tag{1.5}$$

and the matrix of ex-ante probabilities of admission, P, satisfies

$$p_s^r(c) = \begin{cases} (1 - \sum_{t<s} p_t^r - \bar{L}_s - \lambda_{r(s)})^+ & \text{if } \bar{L}_s > -\lambda_{r(s)}; \\ 1 - \sum_{t<s} p_t^r & \text{otherwise,} \end{cases} \tag{1.6}$$

where $(\cdot)^+$ indicates $max(\cdot, 0)$.

Corollary 1.1. *When evaluated at an equilibrium choice density vector, Equation* (1.5)*, and Equation* (1.6) *yield equilibrium lottery cutoffs and probabilities of admission.*

Equation (1.6) follows immediately from Equation (1.5). The probability of admission at s conditional on submitting rank order list r, p_s^r, is the likelihood of having an unadjusted lottery number less than $1 - \sum_{t<s} p_t^r$ and greater than $\bar{L}_s + \lambda_{r(s)}$.

Equation (1.5) and Equation (1.6) must be solved recursively because they depend on each other. The minimum adjusted lottery number necessary for admission at School s, $\bar{L}_s$, depends only on the probability of admission at schools ranked ahead of s, and the probability of admission at School s depends only on $\bar{L}_s$ and the probability of admission at schools ranked ahead of s. The minimum adjusted lottery numbers and the probabilities of admission can, therefore, be computed recursively as follows:

1. Compute $\bar{L}_1$.

2. Use $\bar{L}_1$ to compute p_1^r for all r.

3. Use p_1^r to compute $\bar{L}_2$.

4. Continue until all probabilities of admission and cutoff lottery numbers are computed.

Having characterized the equilibrium, I am now prepared to discuss the Taiwan mechanism's welfare properties. Theorem 1.3 states that the Taiwan mechanism ex ante weakly Pareto dominates deferred acceptance for sophisticated students, which means that every sophisticated student has weakly higher expected utility under the Taiwan mechanism than under deferred acceptance, prior to the realization of the random lottery numbers. The result, which follows from Abdulkadiroglu et al.'s (2011b) proof of the same theorem for the Boston mechanism, shows how the Taiwan mechanism's efficient sorting of students based on the strength of their preferences can benefit everyone who strategizes. Theorem 1.3 is also related to Akbarpour et al.'s (2022) main result, which states that students always prefer manipulable mechanisms if and only if they have an outside option. In my model, all students have the same outside option, so Akbarpour et al.'s (2022) result implies that the Taiwan mechanism weakly Pareto dominates deferred acceptance. The formal definition of ex ante weak Pareto dominance is as follows:

Definition 1.4. *The deduction rule parameterized by λ weakly ex ante Pareto dominates the deduction rule parameterized by λ' for sophisticated students, if for all utility vectors, v, and all equilibria, σ and σ',*

$$v \cdot P^{\sigma(v,\lambda)}(c(\sigma), \lambda) \geq v \cdot P^{\sigma'(v,\lambda')}(c(\sigma'), \lambda').$$

Theorem 1.3. *Any value of $\lambda > 0$ weakly ex ante Pareto dominates $\lambda = 0$ for sophisticated students.*

The intuition is that when $\lambda = 0$, all students reveal their true ordinal preferences, and all students are treated equally, so the students' probabilities of admission are simply the schools' capacity constraints. When $\lambda \neq 0$, any student can achieve those same probabilities of admission by submitting each rank order list with probability c^r. This is because the total number of students a school accepts must equal the average probability of admission for each rank order list weighted by the portion of students who submit that rank order list. The students weakly prefer their equilibrium strategy to achieving the deferred acceptance probabilities because the deferred acceptance probabilities remain in their choice set. Although the result follows from earlier papers, I provide a

proof in the appendix using my notation for clarity.

Theorem 1.3 shows that sophisticated students prefer any deduction rule with $\lambda > 0$ to deferred acceptance. An immediate corollary is that if all students are sophisticated the Taiwan mechanism ex ante weakly Pareto dominates deferred acceptance for all students. It might seem intuitive that deduction rules with higher values of λ weakly Pareto dominate deduction rules with lower values of λ when all students are sophisticated, but this is not the case. The appendix section A.3 presents a class of counterexamples which demonstrates that high deduction rules do not generally weakly Pareto dominate lower deduction rules.

The fact that the Taiwan mechanism weakly Pareto dominates deferred acceptance when all students are sophisticated does little to recommend the Taiwan mechanism when there is strong evidence that many students fail to strategize. The Boston mechanism's potential efficiency advantages over deferred acceptance have been well established for some time, but economists continue to criticize the Boston mechanism because it may benefit sophisticated students at the cost of sincere students. Theorem 1.4, below, shows that, in this setting, the same criticism cannot apply to the Taiwan mechanism. The mechanism designer can always choose λ such that sincere students' aggregate welfare is greater in any Taiwan mechanism equilibrium than under deferred acceptance.

The key intuition is that marginal increases to the deduction rule from the deferred acceptance baseline benefit sincere students at little cost. A marginally positive deduction rule induces sophisticated students who are nearly indifferent between two schools to omit the more competitive option, increasing sincere students' probability of admission to their preferred school. The reduction in sincere students' total probability of admission to the two schools is proportional to λ, and therefore approaches 0 for λ sufficiently small. The deduction rule can be thought of as the cost that sincere students pay to induce sophisticated students to omit preferred schools from their rank order lists. Inducing sophisticated students who are nearly indifferent to switch is effectively free, and therefore improves sincere students' welfare. I now introduce the theorem, and then sketch the proof to elucidate the intuition.

Theorem 1.4. *Suppose f has full support. Then if the norm of the deduction rule, λ, is sufficiently small, sincere students' aggregate welfare is strictly greater in any Taiwan mechanism equilibrium than under deferred acceptance.*

I provide the key intuition for the proof here; the formal proof is in section A.4 in the appendix. The proof demonstrates that the derivative of sincere students' expected utility with respect to λ at $\lambda = 0$ is strictly positive. Recall that all sincere students submit the rank order list $(1, 2, \ldots, N)$, regardless of the value of λ. It follows from Equation (1.6) that sincere students' aggregate utility for λ sufficiently small is

$$\sum_{s=1}^{N} \left[(\bar{L}_{s-1} - \bar{L}_s) - (\lambda_s - \lambda_{s-1})\right]^+]E[v_s]. \tag{1.7}$$

I use Equation (1.5) to plug in for $\bar{L}_s$. When plugging in, I use the fact that for λ sufficiently small, all monotonically increasing rank order lists induce a probability of admission between 0 and 1 at all schools, and are therefore in $\mathcal{R}_2$ by the definition of $\mathcal{R}_2$ (see Equation 1.4). I also use the fact that $1 - \sum_{t<s} p_t^r = \bar{L}_t + \lambda_{r(t)}$, where t is the highest indexed school such that $p_t^r > 0$, and $t < s$. This allows me to rewrite expected utility as

$$\sum_{s=1}^{N} \left[\bar{L}_{s-1} - \frac{\sum_{r:s\in r} c^r(\bar{L}_t + \lambda_{r(t)} - \lambda_{r(s)}) - q_s}{\sum_{r:s\in r} c^r} - (\lambda_s - \lambda_{s-1})\right]^+ E[v_s]. \tag{1.8}$$

In Equation (1.8), $\bar{L}$, c and λ all change as λ changes. Taking the derivative of Equation (1.8), therefore, requires using the product rule, which decomposes the effect of a marginal increase of λ into a mechanical effect and a behavioral effect. The mechanical effect holds c constant and allows $\bar{L}$ and λ to vary, whereas the behavioral effect allows c to vary while holding $\bar{L}$ and λ constant. The mechanical effect on sincere students' probability of admission is 0. At $\lambda = 0$, all students submit the same rank order list: $(1, 2, \ldots, N)$, so the marginal increase in λ changes all students' lottery numbers equally and, therefore, does not change the probabilities of admission to any school for any student. The behavioral effect strictly benefits sincere students. Some students choose to submit a rank order list other than $(1, 2, \ldots, N)$, but because $\lambda = 0$, the deduction rule does not compensate them for skipping some competitive schools. When they omit a school from their rank

order list, they simply exchange their chance of admission at that school for an equivalent probability of admission at a less competitive school. This benefits sincere students who receive the increased probability of admission at a more preferred school in exchange for a decrease in probability of admission at a less preferred school. Sincere students' utility is strictly higher at some $\lambda > 0$ than at $\lambda = 0$ because the mechanical effect of a marginal increase in λ at $\lambda = 0$ is 0, and the behavioral effect is strictly positive.

1.5. Cambridge Application

Theorem 1.4 states that there exists a deduction rule such that the Taiwan mechanism outperforms deferred acceptance for sincere students, but choosing the optimal deduction rule may be difficult. If only a small set of deduction rules outperform deferred acceptance, the mechanism designer is liable to choose a deduction rule that performs poorly, and the Taiwan mechanism should be avoided. This section uses data from the Cambridge Public School system to study the Taiwan mechanism's welfare properties. My results show that the identified equilibrium for a wide variety of deduction rules outperforms deferred acceptance for both sincere and sophisticated students, although sincere students' welfare gains are minimal. The Taiwan mechanism may have other equilibria where the result does not hold, but I do not find any evidence for multiple equilibria. Insofar as other settings are similar to Cambridge, the Taiwan mechanism may provide greater welfare than deferred acceptance even if the mechanism designer fails to select the optimal deduction rule.

This section also demonstrates that the Taiwan mechanism continues to perform well even when two of the assumptions in the model do not hold. In the model, I assume that all students share the same ordinal ranking of schools. In reality, although students' preferences are correlated, they do not share the same ordinal ranking. Table 1.2 displays the mean percentage of students who prefer each school as their first choice, second choice, and third choice from simulations of students' estimated preferences in 2007-2008. Graham Parks, Haggerty, Cambridgeport, and Flet Mayn are the most popular options, but there is no consensus first choice, and even the least popular schools are preferred by some students. The model also assumes that the schools' preferences over the students are solely determined by lottery numbers. In practice, many school districts give priority

Table 1.2: Simulations of Students' Preferred Schools

School	First Choice	Second Choice	Third Choice
Graham Parks	10.22%	16.85%	20.23%
Haggerty	13.91%	9.90%	8.72%
Baldwin	8.43%	5.44%	5.44%
Morse	1.82%	3.76%	4.51%
Amigos	7.42%	8.33%	8.15%
Cambridgeport	10.21%	7.81%	6.89%
King Open	7.39%	7.49%	8.00%
Peabody	7.24%	5.40%	4.63%
Tobin	3.85%	3.05%	3.44%
Flet Mayn	17.76%	18.52%	14.52%
Kenn Long	2.48%	3.40%	4.02%
MLK	7.79%	9.75%	11.10%
King Open Ola	1.47%	0.30%	0.37%

Notes: This table displays the portion of students who prefer each school as their first, second or third choice among all elementary schools in Cambridge. The table is generated from simulations of students' estimated preferences in the 2007-2008 school year.

to students at certain schools based on their distance from the school or whether they have a sibling who attends the school and only use lottery numbers to break ties in priority. Cambridge gives two priority points to students at any school their sibling attends, and one priority point to students at the three schools closest to them. Each school's preference for each student is the sum of the student's priority points at that school and the student's $U[0,1]$ lottery number, which is the same for all schools. I show that an equilibrium of the Taiwan mechanism outperforms deferred acceptance even when students do not share the same ordinal rankings, and schools apriori discriminate between students.

1.5.1. Data

I obtained the same data from the Cambridge Public Schools' Controlled Choice plan for kindergarten admission from academic years 2004-2005 to 2008-2009 that Agarwal and Somaini (2018) used. Table 1.3 largely replicates the summary statistics from Agarwal and Somaini (2018). The schools have slightly more capacity in my data, but I was unable to identify the source of the discrepancy.

Table 1.3: Summary Statistics

Year	2004	2005	2006	2007	2008	Average
			Panel A: District Characteristics			
Schools	13	13	13	13	13	13.00
Seats	473	482	491	509	454	481.80
Students	412	432	397	457	431	425.80
Paid-Lunch	68.45%	62.04%	62.97%	70.90%	67.52%	66.37%
			Panel B: Student's Ethnicity			
White	46.84%	46.76%	45.34%	48.58%	49.19%	47.34%
Black	26.94%	21.99%	23.68%	22.32%	23.20%	23.63%
Asian	16.75%	17.82%	15.11%	13.35%	17.63%	16.13%
Hispanic	9.22%	11.34%	10.08%	8.53%	8.58%	9.55%
			Panel C: Language Spoken at Home			
English	72.33%	73.38%	72.54%	78.12%	81.21%	75.52%
Spanish	3.40%	4.17%	3.53%	3.72%	2.55%	3.47%
Portuguese	0.49%	1.16%	1.01%	0.88%	1.16%	0.94%
			Panel D: School Distance			
Closest School	0.43	0.67	0.43	0.47	0.45	0.49
Average School	1.91	1.93	1.93	1.93	1.89	1.92
			Panel E: Students with Priority at Ranked Schools			
Sibling Priority at 1st Choice	38.11%	33.56%	31.74%	23.85%	33.87%	32.23%
Sibling Priority at 2nd Choice	3.88%	3.01%	0.76%	1.75%	1.62%	2.20%
Sibling Priority at 3rd Choice	0.49%	1.62%	0.76%	0.88%	0.46%	0.84%
Proximity Priority at 1st Choice	52.67%	51.85%	49.87%	50.77%	51.51%	51.33%
Proximity Priority at 2nd Choice	41.50%	34.49%	36.52%	33.04%	36.66%	36.44%
Proximity Priority at 3rd Choice	22.33%	23.84%	23.93%	24.51%	21.11%	23.14%

Notes: This table replicates the summary statistics in Agarwal and Somaini (2018). As in Agarwal and Somaini (2018), the data consists of students participating in the January Kindergarten Lottery. Seats is the total number of seats in the elementary school system available to be allocated through the Controlled Choice Plan. Students is the total number of students enrolled through the Controlled Choice Plan. Paid-lunch status is determined by whether the student applied for the federal lunch subsidy. Closest school is the distance in miles to the school closest to the student and average school is the average distance in miles of all elementary schools in Cambridge. Sibling priority and proximity priority at the ith choice are indicators for whether the student has the relevant priority at the school they rank ith.

Cambridge's mechanism is similar to the Boston mechanism. The main distinction is that Cambridge caps the number of free/reduced-lunch students and paid-lunch students in each school to promote socioeconomic diversity. Schools consider students sequentially based on their preferences for the students and accept a paid-lunch (free/reduced-lunch) student if the school is not yet full and the paid-lunch (free/reduced-lunch) cap has not yet been reached. Students, therefore, have different probabilities of admission depending on their paid-lunch status, and these different probabilities of admission might cause them to prefer different mechanisms. The appendix includes separate estimates of welfare for paid-lunch and free/reduced-lunch students because the mechanism treats them differently.

1.5.2. Estimation of preferences

I use Agarwal and Somaini's (2018) methodology and replication code to estimate p_{soph} and the students' preferences. Each student's preference for each school is assumed to be $N(\hat{\beta} \cdot X, \hat{S})$, where X is a set of student level covariates. The estimation procedure uses a Gibbs sampler to choose $\hat{\beta}$, $\hat{S}$, and p_{soph} to maximize the likelihood of observing the students' submitted rank order lists. For every rank order list, r, there is a set of utility vectors, V^r, such that r is the optimal rank order list for all $v \in V^r$. The estimation uses two sources of variation to identify the probability mass of V^r. Different market conditions across years means that the same r can generate different probabilities of admission, which changes the boundaries of V^r. The difference in the number of students who submit r across years identifies the probability mass of the change in V^r. The second source of variation assumes that distance from schools is additively separable in the students' indirect utility function. If so, variation in distance creates variation in the set of indirect utilities (excluding utility attributable to distance) such that submitting r is optimal. Observing how the likelihood of submitting r changes with respect to distance identifies the probability mass of the change in the indirect utilities. The portion of students whose simulated preferences rationalize their submitted rank order list identifies p_{soph}. Table 1.4 provides the estimates of $\hat{\beta}$ and $\hat{S}$; 62.2% of paid-lunch students and 68.4% of free/reduced-lunch students are sophisticated.

Table 1.4: Estimated Preference Parameters

School	Constant	Paid-Lunch	Sibling	Black	Asian	Hispanic	Other Ethn	Spanish	Portuguese	Other Lang	Unobs. S.D.
Graham Parks	1.85	0.59	3.35	-0.13	0.11	-0.14	-0.30	-0.66	-1.75	-0.04	0.94
Haggerty	2.06	0.34	4.71	-0.48	-0.29	-0.37	0.00	-1.33	-1.17	0.31	1.41
Baldwin	2.11	0.29	2.76	-0.07	0.19	0.07	0.23	-1.00	-0.71	-0.15	0.86
Morse	1.54	-0.03	3.68	0.42	0.43	-0.18	0.33	-0.57	-1.51	0.46	1.21
Amigos	0.53	0.18	10.11	0.06	-0.23	1.18	0.13	0.88	-1.37	-0.89	1.37
Cambridgeport	1.49	0.17	4.54	-0.05	-0.21	-0.08	-0.03	-0.76	-10.67	0.23	1.05
King Open	1.26	0.05	5.52	0.27	0.09	0.06	0.03	-1.01	0.00	-0.10	1.22
Peabody	1.13	-0.22	3.85	0.41	0.34	-0.19	-0.21	-0.15	-1.29	0.01	1.51
Tobin	0.92	-0.76	4.86	0.53	0.59	0.17	-0.13	0.27	-14.69	0.38	1.67
Flet Mayn	0.49	-1.22	3.25	1.06	-0.08	0.49	0.74	0.06	-6.95	0.40	1.54
Kenn Long	1.20	-0.51	2.87	0.03	0.23	0.31	-0.72	-0.10	-10.35	0.29	1.32
MLK	0.45	-0.57	2.64	0.77	0.61	0.50	0.18	0.19	-5.24	0.48	1.37
King Open OLA	-0.67	-0.31	12.41	0.29	-2.09	-2.53	-7.07	-2.86	4.80	-2.00	0.77

Notes: This table presents estimates of the preference parameters for the model with sincere and sophisticated students in Agarwal and Somaini (2018). The excluded ethnicity is white, and the excluded language is English. The distance coefficient is normalized to -1, so all magnitudes are in equivalent miles. N=2128.

1.5.3. Estimating TWM Equilibrium and Welfare

In the Taiwan mechanism counter-factual, School s's preference for student i is the sum of Student i's priority at School s, Student i's lottery number, and the deduction rule corresponding to Student i's ranking of School s. For example, suppose $\lambda_2 = 1$ and student i has proximity priority at School s, a lottery number of .5, and ranks School 2 second. School s's final preference for Student i is $1 + .5 - 1 = .5$. If Student j has no priority at School s, but ranks School s first and has a lottery number greater than .5, School s would admit Student j ahead of Student i. If Student j's lottery number is less than .5, School s prefers Student i.

In equilibrium, sincere students submit their true preferences, and sophisticated students submit the rank order list that maximizes their utility. Sophisticated students' optimal strategies depend on the probabilities of admission, but the probabilities of admission depend on the students' strategies. I, therefore, iterate over probabilities of admission to find a fixed point, which is an equilibrium of the Taiwan mechanism. An added challenge is that the probabilities of admission for a given simulation are random variables that depend on the random draws of students' utility and lottery numbers. I address this challenge by estimating the probabilities of admission many times for each iteration of the students' strategies, and setting the probabilities of admission for the next step in the iteration equal to the mean.

The algorithm to estimate equilibrium welfare proceeds as follows:

1. Initialize a matrix of probabilities of admission, P^0.

2. Perform each of the following steps N times:

   (a) Draw utilities for all students from the distribution of utilities, $N(\hat{\beta} \cdot X, \hat{S})$, and draw an indicator for whether each student is sophisticated.

   (b) Compute students' choices assuming P^0 represents the true probabilities of admission. Sophisticated students submit the optimal rank order list, which maximizes their utility from attending each school multiplied by their probability of admission under P^0. Sincere

students submit their true preferences.

(c) Subtract λ_k from each student's priority at the school they rank kth.

(d) Run deferred acceptance using the students' choices computed in step 2.b and the priorities computed in step 2.c and record $\bar{L}$, the minimum lottery numbers necessary for admission at the schools.

(e) Use Equation A.5 to compute P from $\bar{L}$. The equation generalizes Equation 1.6 for the case where students may have priority at some schools.

3. Set P^1 equal to the mean P from all N runs of step 2.

4. Repeat steps 2-3 until P converges to P^*. Convergence is considered achieved when the maximum change in probability of admission to a school between iterations for any priority type, rank order list combination is less than .005. I also require that the algorithm complete at least 15 iterations,

In my main set of results, I initialize P^0 as P^* of a nearby deduction rule. As a robustness check, I also use different initializations of P^0 to search for multiple equilibria. I used a high specification, which adds .2 to P^0, and a low specification, which subtracts .2 from P^0. Table A.13, in the appendix, shows that all initializations converge to nearly identical probabilities of admission.

The set of possible rank order lists is extremely large, so computing the optimal strategy for sophisticated students can be computationally infeasible. I employ two different strategies to allow for efficient computation of sophisticated students' optimal strategies. My first strategy is to restrict attention to Taiwan mechanisms with constant λ, i.e. $\lambda = (0, \lambda, \ldots, \lambda)$. Computing best responses for constant deduction rules is easier because constant deduction rules only give students an advantage at the school they rank first. The students, therefore, choose the first school strategically, but subsequently reveal their preferences in the correct order (Lemma A.10). To illustrate this, consider a student who ranks their second-favorite school first, and chooses between ranking their favorite and third-favorite school second. Regardless of whether the student ranks

their third-favorite school second or third, the student faces the same deduction at their third-favorite school. Therefore, excluding a student's first-choice, ranking a less preferred school ahead of a more preferred school simply transfers probability of admission from the more preferred school to the less preferred school, which harms the student. The upshot is that I only need to consider rank order lists which truthfully reveal preferences after the student's first choice, so there are only as many possible lists as there are schools. It is computationally feasible to compute the utility from each strategy and select the maximum. If λ were not constant, then a much larger set of strategies becomes plausible, and it is no longer feasible to determine which strategy maximizes utility.

My second strategy prohibits students from submitting rank order lists with greater than 3 schools. This restriction significantly reduces the set of possible strategies so that computing students' best responses remains feasible even when λ is not constant. I use this strategy to estimate welfare for deduction rules that are not constant. Many public school districts actually do limit the length of rank order lists, and CPS only allows rank order lists with 3 schools or fewer.

After computing the equilibrium P^*, I let each student best respond to P, compute the matching, and record average welfare. The following steps summarize the procedure:

5. Perform each of the following steps O times:

   (a) Use P^* to repeat steps 2a-2d and record the final matching.

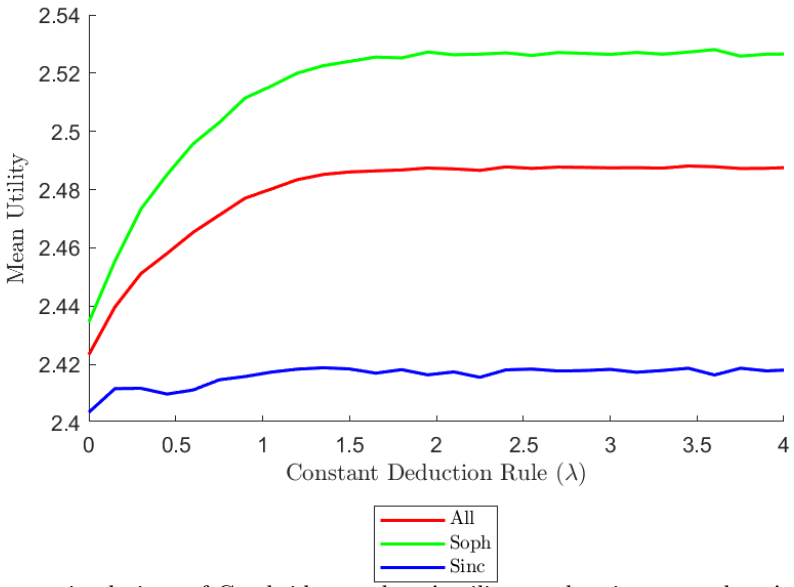
   (b) Record average utility at the matching computed in step 5a.

6. Estimated welfare for the Taiwan mechanism with deduction rule λ is the average of welfare across the O draws from step 5.

First, I employ the strategy which restricts to constant deduction rules. Figure 1.2 presents welfare for all students, sincere students, and sophisticated students as a function of λ for the 2007 matching market, where the deduction rule is $\lambda = (0, \lambda, \ldots, \lambda)$. The figure plots welfare for $\lambda \in [0, 4]$ because the mechanism is equivalent for all $\lambda > 4$. Once $\lambda > 4$, all students who apply in the first-round are admitted before students who apply in subsequent rounds because the maximal priority

points (3) plus lottery number (1) equals 4. Note, however, that the constant deduction rule does not converge to the Boston mechanism as λ increases because students who apply in rounds other than the first-round are treated the same as students who apply in later rounds. The graph demonstrates that, at the identified equilibria, higher values of λ significantly increase sophisticated students' welfare. The gains for sincere students are modest, but their utility is higher at all $\lambda > 0$ than at $\lambda = 0$. Sophisticated students have slightly higher utility even when $\lambda = 0$ because they are slightly more likely to be free/reduced-lunch and free/reduced-lunch students face a less competitive market. Equivalent figures for the remaining years in my data, and figures that present welfare estimates separately for paid-lunch and free/reduced-lunch students are in the appendix in section A.8. All the figures other than 2004 support the same conclusion. In 2004, sincere students' utility peaks when λ is between .3 and .6. At higher values of λ, it falls slightly below its value at $\lambda = 0$.

Figure 1.2: Estimated Utility in 2007 for Constant Deduction Rules



Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

I use the second strategy, which restricts rank order lists to include at most three schools, to study deduction rules which are not constant. Table 1.5 displays students' utility at different

Table 1.5: Estimated Utility in 2007 for Variable Deduction Rules

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.43	2.44	2.45	2.45	2.45	2.45
	$\lambda_2 = .4$	2.46	2.46	2.47	2.47	2.47	2.47
	$\lambda_2 = .8$	2.47	2.47	2.47	2.47	2.47	2.47
	$\lambda_2 = 1.2$	2.48	2.48	2.48	2.48	2.48	2.48
	$\lambda_2 = 1.6$	2.48	2.48	2.48	2.48	2.48	2.48
	$\lambda_2 = 2$	2.48	2.48	2.48	2.48	2.48	2.48
Sincere	$\lambda_2 = 0$	2.40	2.40	2.41	2.41	2.41	2.41
	$\lambda_2 = .4$	2.40	2.40	2.40	2.40	2.40	2.40
	$\lambda_2 = .8$	2.40	2.40	2.40	2.40	2.40	2.40
	$\lambda_2 = 1.2$	2.41	2.40	2.40	2.40	2.40	2.40
	$\lambda_2 = 1.6$	2.40	2.40	2.40	2.40	2.40	2.40
	$\lambda_2 = 2$	2.40	2.40	2.40	2.40	2.40	2.40
Sophisticated	$\lambda_2 = 0$	2.45	2.47	2.47	2.48	2.48	2.48
	$\lambda_2 = .4$	2.49	2.50	2.50	2.50	2.51	2.51
	$\lambda_2 = .8$	2.51	2.52	2.52	2.52	2.52	2.52
	$\lambda_2 = 1.2$	2.52	2.52	2.52	2.52	2.52	2.52
	$\lambda_2 = 1.6$	2.52	2.52	2.52	2.53	2.52	2.52
	$\lambda_2 = 2$	2.53	2.52	2.53	2.53	2.53	2.53

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

deduction rules. The rows hold λ_2 constant, and the columns hold $\lambda_3 - \lambda_2$ constant, which is the advantage students who apply in the third round have over students who apply in the second round. The analysis verifies that, in the identified equilibrium, a wide range of deduction rules improve sophisticated students' welfare without harming sincere students. The results for the remaining years, and heterogeneous utility estimates by free/reduced-lunch status, are in section A.8.

1.6. Conclusion

This paper demonstrates that, under the assumptions in the model, there is always a Taiwan mechanism that outperforms deferred acceptance for sincere students, and shows in an empirical setting where the model's assumptions do not hold, that a wide range of deduction rules outperform deferred acceptance. My results build on Abdulkadiroglu et al. (2011b). They show that the Boston mechanism *may* benefit sincere students, and I demonstrate that the Taiwan mechanism *necessarily* benefits sincere students. Furthermore, I demonstrate that the Taiwan mechanism may provide

substantially greater aggregate welfare for sincere students than both the Boston mechanism and deferred acceptance. If we take the sincere and sophisticated students binary seriously, my results show that the optimal mechanism for sincere students is often a Taiwan mechanism with a deduction rule that compromises between the deferred acceptance and Boston mechanism extremes.

Of course, the sincere and sophisticated students model is limited, and better models of the mistakes students make playing manipulable mechanisms would facilitate better welfare comparisons between manipulable mechanisms and deferred acceptance. Kapor et al. (2020) and De Haan et al. (2023) have made admirable progress by demonstrating that there are students who do not fit into the sincere and sophisticated students binary; they neither submit their true preferences, nor the optimal rank order list. However, the literature does not yet have an accepted model for these mistakes, and welfare comparisons of the Taiwan mechanism and deferred acceptance will likely depend on how we model these errors. This paper demonstrates the need for such a model to better understand the benefits of deferred acceptance and to accurately evaluate alternatives.

In the meantime, we may speculate about the Taiwan mechanism's performance if students suffer from different biases identified in the literature. For example, the Taiwan mechanism will likely perform well if the mistakes are caused by correlation neglect, which Rees-Jones et al. (2020) demonstrates may be present in school choice mechanisms. The Taiwan mechanism can protect a student who does not realize that conditional on rejection from their first choice they have little chance of admission at their second choice by making $\lambda_3 - \lambda_2$ small. On the other hand, the Taiwan mechanism cannot help students who place their top choice near the bottom of their rank order list because they falsely believe they have little chance of admission at their top choice. These students are likely to be admitted to one of the schools they ranked ahead of their top choice, in which case the deduction they face at their top-choice school will be irrelevant.

CHAPTER 2

BROAD VALIDITY OF THE FIRST-ORDER APPROACH IN MORAL HAZARD

Authors: Eduardo Azevedo and Ilan Wolff.

Unique contribution: I wrote all the code and performed all numerical analysis. The proofs were joint work. The manuscript was also joint work, but I focused on section 5.2 and the appendix on functional form derivations.

2.1. Introduction

One of the workhorse models in economics and related fields is the principal-agent problem with moral hazard. The principal hires the agent to take an action a in $\mathbb{R}_+$ that affects the distribution of output y. The principal can only condition payments on realized output, and designs a contract $w(y)$ to provide incentives to the agent. The agent chooses the action a to maximize her utility from wages minus her cost of effort,

$$\mathbb{E}\left[u(w(y))|a\right] - c(a),$$

and the principal chooses the contract to maximize expected profits.

The main solution method for this problem is the first-order approach, which assumes that only local deviations in a are binding. This yields a tractable, calculus-based formula for the optimal contract (see Holmström (1978) and the excellent survey Georgiadis (2022)). Many results in the literature rely on the first-order approach, even though Mirrlees (1999) (circulated in 1975) showed it is not always valid. In applied work, restrictive assumptions such as linear contracts or binary effort are often made to avoid this issue.[2]

[2]Examples of papers assuming the first-order approach include, to cite a few, Jewitt et al. (2008); Moroni and Swinkels (2014); Chaigneau et al. (2022); Castro-Pires et al. (2024). Virtually all early work, including the seminal papers by Holmström (1978, 1979), and Zeckhauser (1970) assumes it. Holmström (1979) notes that "one has to assume that the agent's optimal choice of action is unique for the optimal [...] This assumption seems very difficult to validate [...] and regrettably we have to leave the question about its validity open." Edmans et al. (2009) is an example of an important recent paper using both binary effort and linear contracts. See the surveys Bolton and Dewatripont (2005); Salanié (2005); Georgiadis (2022) for further references.

Unfortunately, existing sufficient conditions for the first-order approach are restrictive. The seminal papers are Rogerson (1985) and Jewitt (1988), followed by an extensive literature.[3] (Kadan et al. (2017)) summarize the general view that "conditions facilitating the first-order approach are typically quite demanding". The key issues are elegantly explained by Chaigneau et al. (2022) and Conlon (2009). They explain that the sufficient conditions require contracts to be close to concave, precluding common contracts such as compensation with stock options.[4]

Our main result, Theorem 2.1, shows that the first-order approach is broadly valid, as long as the agent's reservation utility is sufficiently high. The result holds even when contracts are option-like and the agent's problem has multiple local maxima. We consider a setting with limited liability, because of its practical importance and because it is known that limited liability or a similar assumption is needed to guarantee existence (Moroni and Swinkels, 2014). We find that, for low reservation utility, the standard view in the literature holds and the first-order approach often fails. However, for high reservation utility, we show that the first-order approach is satisfied. Thus, the issues with the validity of the first-order approach are likely to be important in examples with low reservation utility like when a monopsonist principal can impose highly unfavorable contracts on agents. When reservation utility is low, the first order approach is often invalid because it produces contracts where the agent has a global deviation to exert minimal effort. However, the first order approach is likely to be valid in settings with high reservation utility, such as competitive labor markets with high productivity workers. As reservation utility goes up, the first order approach's contract increases the incentive to work, eliminating global deviations, and restoring the first-order approaches validity.

Theorem 2.1 also shows that the optimal contract exists, is unique, and characterized by a

[3]Important generalizations include Sinclair-Desgagné (1994); Conlon (2009); Jung and Kim (2015); Chade and Swinkels (2020), and Jung et al. (2024). Chaigneau et al. (2022, 2024) develop sufficient conditions with limited liability.

[4]Conlon (2009) explains the intuition clearly: "Unfortunately, the Jewitt conditions are tied to the concavity, not only of the technology, but also of the payment schedule. Thus, there are interesting cases where the Jewitt conditions should fail, since the payment schedule is often not concave. For example, managers often receive stock options, face liquidity constraints, [...] I believe that it is natural to expect the first-order approach itself to fail in such cases, since the agent's overall objective function will tend to be nonconcave [...]. The fact that CDFC and CISP imply concavity of the agent's payoff, regardless of the curvature of [the payment], then suggests that the CDFC/CISP conditions are very restrictive, not that the first-order approach is widely applicable.".

simple formula. Section 2.6 show that optimal contracts are piecewise linear option contracts for many standard distributions and log agent utility, and gives simple computational algorithms. Thus, the limited liability model seems well suited for applied theory and empirical models that seek to predict the shape of optimal contracts $w(y)$.

Before going into the analysis, section 2.2 explains our results through simple examples, and reconciles our findings with the literature. Section 2.3 gives definitions and section 2.4 states Theorem 2.1. Section 2.5 outlines the proof. Section 2.6 considers applications, extensions, and counterexamples.

2.2. Examples and Literature Review

2.2.1. Examples Illustrating the Main Result

Consider the following **Gaussian-log utility example**. A risk-neutral principal hires an agent to work. The agent chooses action a at a cost $c(a)$ proportional to a^2. Output y, which accrues to the principal, is normally distributed with mean a and standard deviation σ. The agent has some independent assets and log utility, so that her utility from a payment of $x \geq 0$ is $u(x) = \log(w_0 + x)$. For concreteness, suppose the principal is the owner of a small company and the agent is a professional manager. Illustrative values are a choices in the ballpark of \$100,000, standard deviation σ of \$20,000, and w_0 equal to \$50,000. The agent has reservation utility $\bar{U}$.

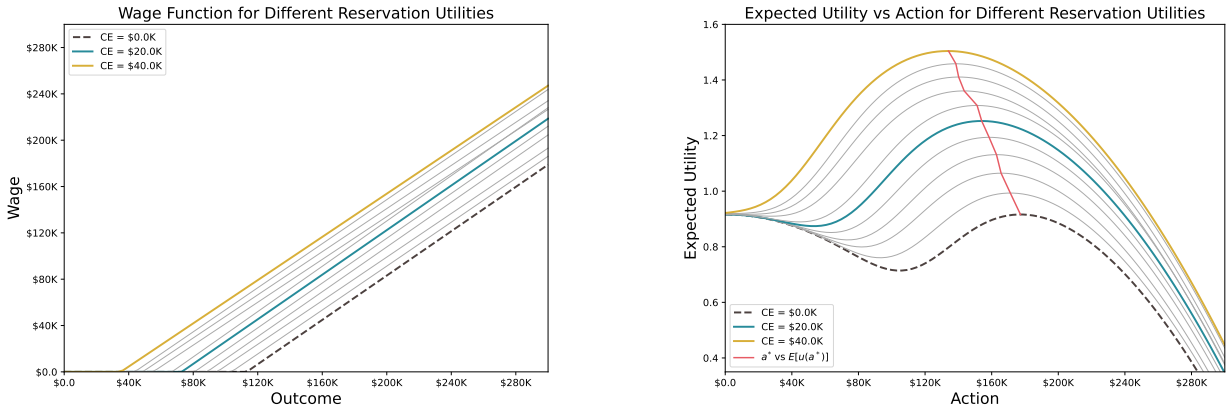
The principal designs a compensation contract with wage $w(y)$ to maximize profits. The optimal contract balances the goals of inducing effort, reducing the agent's risk, and providing the agent with her reservation utility. It is known that, without limited liability, an optimal contract does not exist. Mirrlees (1999) showed that it is possible for the principal to get arbitrarily close to her perfect-information payoff using contracts that impose harsh punishments with small probability. Thus, we consider the case of limited liability, requiring $w(y) \geq 0$.[5]

Figure 2.1 plots optimal contracts for a range of reservation utilities $\bar{U}$, with certainty equivalents

[5]Moroni and Swinkels (2014) show that, without limited liability, existence often fails if the agent has negative infinite utility from a finite wage. Limited liability is a common and minimal way to avoid these types of existence issues.

from \$0 to \$40,000. The left panels plot the optimal wage schedules $w(y)$. The optimal contracts are intuitive. They have a flat region to the left where, due to limited liability, the agent is paid $w(y) = 0$, and an increasing region to the right, which gives the agents incentives to work. As reservation utility increases, contracts become more generous. Wages schedules shift up, and the minimum output required to receive positive pay moves to the left.

Figure 2.1: Optimal Contracts with Gaussian Distribution and log Utility



(a) Canonical Contracts

(b) Expected Utility

Note: Panel a displays the wage function that solves the principal's profit maximization problem at different values of $\bar{U}$, and Panel b displays the agent's expected utility from a grid of possible actions given the corresponding wage functions from Panel a. The red line in Panel b plots the optimal action against optimal utility for the agent. The values of $\bar{U}$ are expressed in terms of certainty equivalents. The agent's utility is $u(x) = log(\frac{x}{20K})$, the cost function is $c(a) = \theta a^2$, where $\theta = .01$, and the agent's starting wealth is $w_0 = 50K$. The output is $y \sim N(a, (20K^2))$, where a is the agent's action. Dotted lines indicate that the FOA is invalid. Our optimization routine has only one global IC constraint at $a = 1e-3$. We conservatively say that the FOA is invalid if the Lagrange multiplier on the global IC constraint is greater than $1e-5$ or if deviating to an action between 0 and .1 reduces the agent's utility by less than $1e-2$.

The right panel is crucial for understanding when the first-order approach is valid. It plots the agent's expected utility $U(v^*, a)$ as a function of a, given the optimal contract. The first-order approach fails whenever there is a binding non-local deviation. In these cases, solving the problem with only the local incentive constraint leads to incorrect solutions, as pointed out by Mirrlees (1999). Cases in which the first-order approach fails are denoted with dashed lines.

The first-order approach fails at the lowest reservation utility, with certainty equivalent of \$0. The optimal contract is a tough bargain. The agent is expected to exert effort a equal to $\$177.2K$.

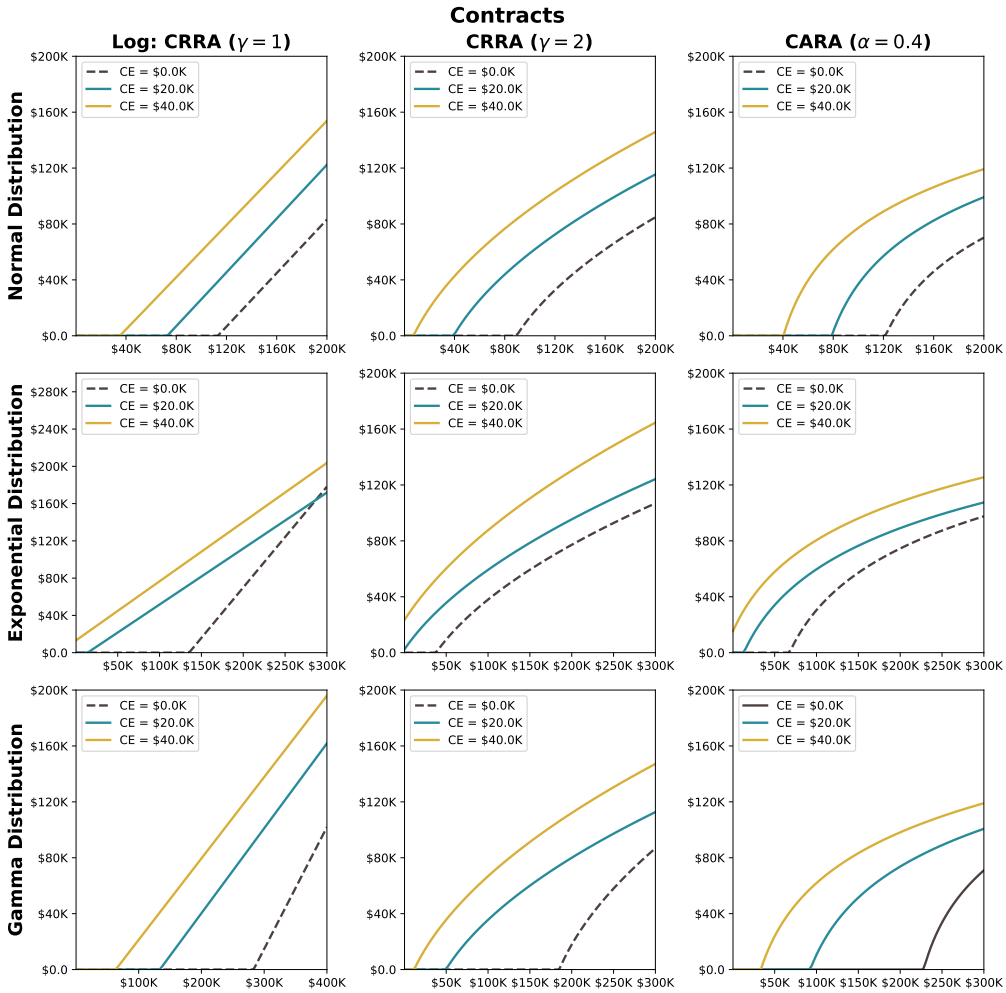
The contract $w(y)$ specifies a payment of zero for any output lower than $\$113.1K$. Thus, the agent needs to produce relatively high output to get paid at all. This contract induces two local maxima in the function $U(v^*, a)$. The agent is indifferent between basically giving up with a local maximum at $a \approx 0$ and a local maximum at the recommended effort level $a = \$177.2K$. This is consistent with the standard view in the literature. Option-like contracts are natural, and thus it is difficult to find general sufficient conditions under which the first-order approach holds.

Surprisingly, the first-order approach becomes valid as soon as reservation utility is slightly higher. The constraint of giving the agent higher utility forces the principal to offer contracts with a higher value of $U(v^*, a)$ at the recommended action. This constraint moves the rightmost local maximum of $U(v^*, a)$ up. The rightmost maximum goes up faster than the leftmost local maximum. Thus, the rightmost maximum becomes the unique global maximum. Numerically, the first-order approach is valid for any reservation utility with certainty equivalent above \$600, as seen in the figure.

Theorem 2.1 shows that this example is more general. The theorem shows that, under certain conditions, the first-order approach is always valid for sufficiently high reservation utility. The most substantial condition is that greater output is evidence of greater effort (monotone likelihood ratio). Figures 2.2 and 2.3 illustrate the generality of this point by plotting the same graphs in examples with different risk preferences and output distributions. All the examples display the same qualitative pattern, consistent with Theorem 2.1. We stress, however, that the monotone likelihood ratio assumption is crucial. For example, if output has a fat-tailed Student-t distribution, there are examples where the first-order approach fails even for high reservation utilities. The reason is that positive and negative outliers are not very informative about effort, so the agent is not penalized for very poor outcomes which generates global deviations to little effort.

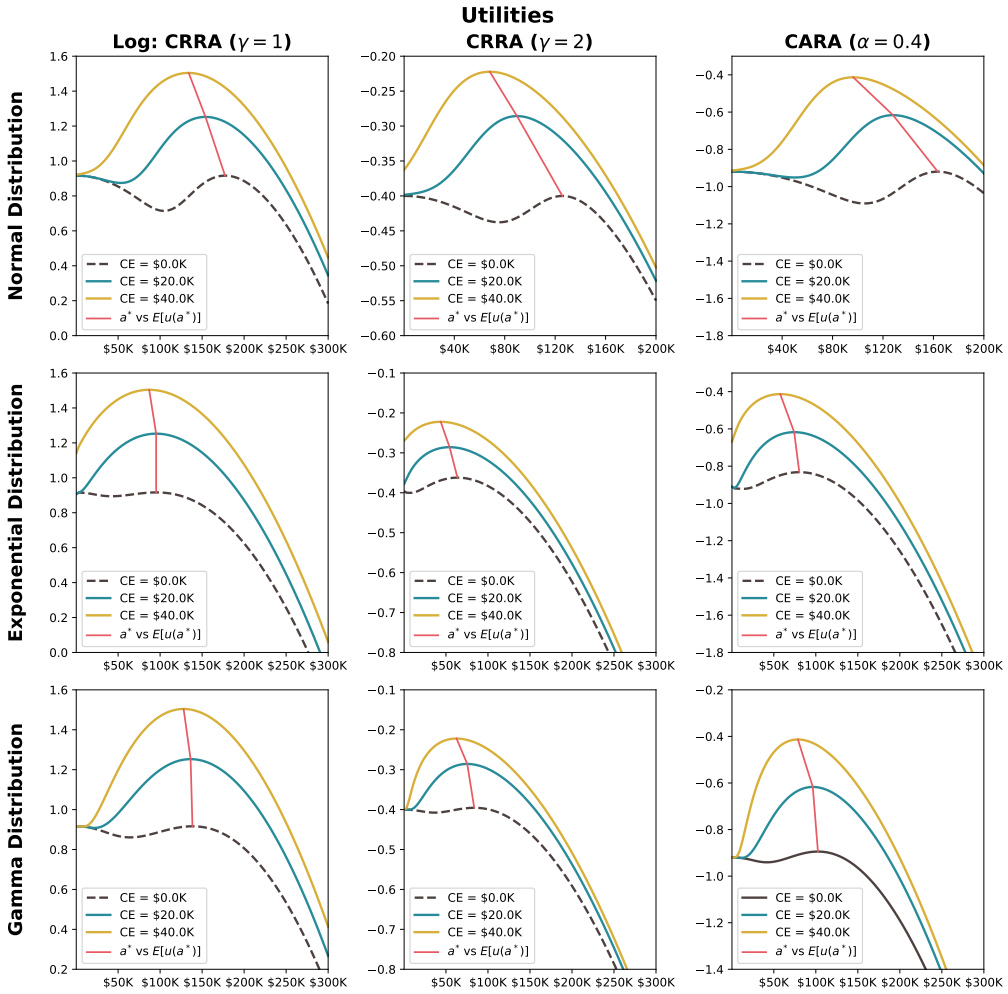
Theorem 2.1 also guarantees existence and uniqueness of an optimal contract. Moreover, the solution has a simple calculus-based formula, as in the classic first-order approach literature following

Figure 2.2: Optimal Contracts under Different Distributions and Preferences



Note: This figure displays the wage function that solves the principal's profit maximization problem at different values of $\bar{U}$ for a grid of specifications. The log utility function is $u(x) = log(\frac{x}{20K})$. The CRRA utility function is $u(x) = \frac{1}{1-\gamma} \cdot \frac{x}{20K}^{1-\gamma}$, and $\gamma = 2$. The CARA utility function is $\frac{1}{\alpha} \cdot -\exp(-\alpha \frac{x}{20K})$, and $\alpha = \frac{2}{5}$. The normal distribution is $y \sim N(a, (20K^2))$. The exponential distribution's rate parameter is $\frac{1}{a}$. The gamma distribution has shape parameter $n = 2$, and scale parameter a. Formulas for all utility functions and output distributions are in the online appendix. The cost function is always $c(a) = \theta a^2$, but $\theta = .01$ in the log utility examples, $\theta = .004$ in the CRRA examples and $\theta = .008$ in the CARA examples. Our optimization routine has only one global IC constraint at $a = 1e-3$. We conservatively say that the FOA is invalid if the Lagrange multiplier on the global IC constraint is greater than $1e-5$ or if deviating to an action between 0 and .1 reduces the agent's utility by less than $1e-2$.

Figure 2.3: Agent Utility versus Action under Different Distributions and Preferences



Note: This figure displays the agent's expected utility from a grid of possible actions given the corresponding wage functions from figure 2.2. The specifications are the same as in figure 2.2.

Holmström (1978). The theorem shows that the optimal wage equals[6]

$$w(y) = k \circ g\left(\lambda + \mu S(y|a_0)\right). \tag{2.1}$$

The function $k \circ g$ is determined by the utility function and limited liability constraint. $S(y|a_0)$ is the score function (also known as the likelihood ratio), determined by the distribution of output. The optimal action is a_0, and λ and μ are Lagrange multipliers.

Theorem 2.1 makes our limited liability setting particularly amenable to applications. As long as one considers a competitive setting with sufficiently high reservation utility, the main difficulties are resolved. Optimal contracts exist, are unique, and can be calculated trivially from equation (2.1).

Section 2.6 provides results to help researchers use the limited liability model. We provide simple calculus formulae for $k \circ g$ and S covering a wide set of examples. These can be readily applied in both theoretical and numerical models. We provide accompanying code for numerical solutions.

An immediate corollary of equation (2.1) is that piecewise linear contracts are optimal in a number of examples. Piecewise linear contracts are options, with a constant value up to a point, and increasing thereafter. The reason is the following. The function $k \circ g$ is piecewise linear when utility is log, and the score function is linear for many common statistical distributions, including Gaussian, exponential, Poisson, and Gamma. More generally, any distribution in the exponential family with linear sufficient statistic has a linear score function. This can be seen in figures (2.1) and (2.2). Optimal contracts are slightly nonlinear for very low reservation utilities, but piecewise linear as soon as the first-order approach starts holding. This extends previous results that rationalize option-like contracts (Innes, 1990; Jewitt et al., 2008; Chaigneau et al., 2024) and contributes to

[6]These additional results are simple given the validity of the first-order approach. Our analysis and results are close to those in Jewitt et al. (2008), albeit we cannot simply use their results due to slightly different assumptions. For example, they assume the Rogerson (1985) conditions, which rule out examples such as the Gaussian distribution of output. Hence, we cannot simply use their results, although our proofs follow broadly similar lines.

the literature rationalizing linear contracts (Holmstrom and Milgrom, 1987; Carroll, 2015).

2.2.2. Relationship to the Literature

At first glance, our results appear to contradict the standard view in the literature that the first order approach requires stringent conditions. In fact, the two are perfectly consistent. Consider the Gaussian example in figure (2.1). When reservation utility is low, the optimal contract produces two distinct local maxima of the agent's utility, violating the first-order approach because the agent then has a profitable non-local deviation. This is exactly the kind of global incentive problem highlighted by Mirrlees (1999). Likewise, the first-order approach fails in all the examples in figure (2.2). Hence, if one insists that the first-order approach must hold *for every reservation utility*, many interesting examples must be ruled out, matching the standard view in the literature.

Our contribution is to show that *if the reservation utility is sufficiently high*, those very same examples do satisfy the first-order approach. The core idea is that requiring the principal to give the agent greater overall utility elevates the payoff at the principal's intended action, pushing the local maximum at the intended action above other local maxima.

Historically, there are three generations of contributions to the first-order approach. The first generation was started by Mirrlees (1999) in the mid 1970s, who showed that the first-order approach does not hold generally. Prior to Mirrlees (1999), the first-order approach was simply assumed with little justification. Subsequent papers then became aware that they had to assume validity of the first-order approach (Holmström, 1978), and developed results that do not depend on the first-order approach (Grossman and Hart, 1983; Araujo and Moreira, 2001).

The seminal papers in the second generation are Rogerson (1985) and Jewitt (1988). They provided sufficient conditions for the first-order approach. Rogerson's conditions are well known to be strict, ruling out many natural distributions, including the Gaussian and all examples in figure (2.2). Jewitt's conditions are more general, and include many interesting cases. Jewitt's main assumption is implicit, in that he requires his conditions to hold for all output levels, in particular ruling out limited liability (Chaigneau et al., 2022). Restrictions similar to limited liability are

known to be important to guarantee existence (Moroni and Swinkels, 2014; Jewitt et al., 2008). In many examples that satisfy the Jewitt conditions, an optimal contract does not exist. This includes the Gaussian - log utility example without limited liability. The second-generation papers have been influential, and much of the theoretical and applied work to this day assumes these conditions.

Our own results are close to Jewitt (1988). We impose similar conditions on the part of the contract where limited liability does not bind, while allowing for limited liability. In contrast, Jewitt (1988) requires the conditions to hold globally, ruling out limited liability, and sometimes creating existence issues (Jewitt et al., 2008; Moroni and Swinkels, 2014). The proof of Theorem 2.1 shows that, with high reservation utility, the limited liability constraint binds with low probability. We then show, in similar lines to Jewitt (1988), that the agent's problem becomes concave.

Our result is applicable because, in many examples, the first-order approach starts holding for relatively modest reservation utility, even when there are multiple local maxima. This can be seen in figures (2.1) and (2.3). The intuition is that, if reservation utility is very high, then $U(v^*, a)$ is concave in a. As reservation utility increases, the functions $U(v^*, a)$ have to unwind into a function with a single peak. And the recommended action becomes optimal well before the function becomes concave. In this sense, our result has a limitation similar to other asymptotic results, such as the central limit theorem, that only holds for large enough sample sizes.

Finally, there is a third generation of work extending Jewitt's single-peakedness approach to multidimensional actions and richer information structures (e.g. Conlon, 2009; Jung and Kim, 2015; Chade and Swinkels, 2020; Chaigneau et al., 2022; Jung et al., 2024). We believe our methods can be adapted to such environments, but we do not pursue that here.

2.3. Model

2.3.1. Model

A risk-neutral **principal** hires an **agent** with limited liability. The agent has utility $u(x) - c(a)$ of receiving a **payment** $x \in \mathbb{R}_+$ and taking **action** a in $\mathcal{A} \subseteq \mathbb{R}_+$. Both u and c are strictly increasing. **Output** $y \in \mathbb{R}$ depends on the agent's action and is distributed according to the **output density**

$f(y|a)$.

A **contract** is a function $v : \mathbb{R} \to u(\mathbb{R}^+)$, which specifies the agent's utility $v(y)$ from the payment as a function of output y. Contracts are defined in terms of utility to simplify notation. To define the wage payments, let the **compensation cost function** k be the inverse of u. A **wage function** is a function $w : \mathbb{R} \to \mathbb{R}^+$. The wage function associated with contract v is $w(y) = k(v(y))$. Let $\mathcal{C}$ be the **set of feasible** contracts.

The **agent's utility from a contract** v given action a is

$$U(v, a) := \int v(y) f(y|a)\, dy - c(a).$$

The **expected wage** is

$$W(v, a) := \int k(v(y)) f(y|a)\, dy.$$

The **cost minimization problem** given an **intended action** $a_0 \in \mathcal{A}$ and **reservation utility** $\bar{U}$ is to choose a contract v to minimize the expected wage subject to the individual rationality (IR) and global incentive compatibility (GIC) constraints. Individual rationality requires that the agent's expected utility is at least reservation utility, and global incentive compatibility requires that choosing a_0 is optimal for the agent. The cost minimization problem is to choose v in $\mathcal{C}$ to

$$\begin{aligned} \text{minimize} \quad & W(v) \\ \text{subject to} \quad & U(v, a_0) \geq \bar{U}, && \text{(IR)} \\ & U(v, a_0) \geq U(v, \hat{a}) \quad \forall \hat{a} \in \mathcal{A}. && \text{(GIC)} \end{aligned}$$

The **relaxed cost minimization problem** replaces the global incentive compatibility con-

straint with the local incentive compatibility constraint (LIC):

$$\begin{aligned} \text{minimize} \quad & W(v) \\ \text{subject to} \quad & U(v, a_0) \geq \bar{U}, && \text{(IR)} \\ & \partial_a U(v, a_0) = 0. && \text{(LIC)} \end{aligned}$$

2.3.2. Assumptions and Notation

We state here the assumptions needed for our results. This section can be skimmed on a first reading.

Assumption 2.1 (Regularity of utility, cost and density)**.** *The set of feasible actions $\mathcal{A} \subseteq \mathbb{R}_+$ is a compact interval including 0. The utility function u is strictly concave and smooth, and has $\lim_{x\to\infty} u'(x) = 0$. The cost function c is strictly convex and smooth, and $c'(0) = 0$. The output density $f(y|a)$ is smooth in the action, a, for all outputs, y.*

We now make technical assumptions that allow us to use Leibniz's rule of differentiation under the integral sign. For this, we impose restrictions on k, f and on the set of feasible contracts. At the same time, we require the set of feasible contracts to be rich enough so that relevant contracts are not ruled out by assumption.

Assumption 2.2 (Regularity of feasible contracts)**.** *The output density function $f(y|a)$ satisfies:*

1. *Every feasible contract is measurable.*

2. *Given any a_1 in $\mathcal{A}$ and a feasible contract v, there exists an integrable function $\theta(y)$ and a neighborhood of a_1 such that, for all a in this neighborhood,*

$$|v\partial_a^n f(y|a)| \leq \theta(y)$$

for $n = 0, 1, 2$ and for all y in the support of $f(\cdot|a)$.

3. *For all a in $\mathcal{A}$ and canonical contract v (as defined in definition 2.2),*

$$\Pi(v, a) < \infty.$$

4. *The set of feasible contracts is convex and includes all canonical contracts.*

We now consider substantive restrictions.

Assumption 2.3 (Assumptions on the distribution of output)**.** *The output density function $f(y|a)$ satisfies:*

1. *For all a, the support of $f(y|a)$ is $\mathbb{R}$.*

2. *$F_a(y|a) < 0$ for all y and a.*

3. *For all a, the score $\partial_a \log f(y|a)$ is strictly increasing in y and its image is $\mathbb{R}$ (this is known in the literature as the monotone likelihood ratio property or MLRP).*

4. *There exists y_0 in $\mathbb{R}$ such that, for all $y \leq y_0$ and a,*

$$f_{aa}(y|a) > 0.$$

Note that we assume that the support of f is $\mathbb{R}$. The numerical examples, like the exponential distribution, suggest that this assumptions is not necessary. We hope to weaken the assumption in the next version of this paper.

Assumption 2.4 (Concavity and limit of inverse marginal utility)**.** *The function k'^{-1} is strictly concave and*

$$\lim_{z \to \infty} z \frac{d}{dz} k'^{-1}(z)$$

is finite.

The function k'^{-1} is the same as Jewitt's (1988) $\omega(z)$. So our assumption that k'^{-1} is strictly concave corresponds to his assumption (2.12).

For some results, we restrict attention to intended actions where it is possible to satisfy the local incentive compatibility constraint.

Definition 2.1. *An intended action a is regular if it is in the interior of $\mathcal{A}$ and there exist a feasible contract v such that*

$$\partial_a U(v, a) \geq 0.$$

2.4. Main Result

We can now state our main result.

Theorem 2.1. *[Validity of the First Order Approach with High Reservation Utility] Given a regular intended action $a_0 > 0$, there exist reservation utilities U^* in $\mathbb{R}$ and $\bar{U}_R \in (U^*, \infty]$ such that:*

1. *For $\bar{U} \geq \bar{U}_R$, neither the relaxed cost minimization problem nor the cost minimization problem are feasible.*

2. *For $\bar{U} \leq \bar{U}_R$, the relaxed cost minimization problem has an almost everywhere unique solution given by proposition 2.1.*

3. *For $U^* \leq \bar{U} < \bar{U}_R$, this is also the almost everywhere unique solution of the cost minimization problem.*

2.5. Proof of the Main Result

The proof of the main result follows from two key propositions. Proposition 2.1 characterizes the solution of the relaxed problem. Proposition 2.2 shows that, for sufficiently high reservation utility, the agent's problem is concave. We now state these propositions and explain the key steps in the argument. Appendix B.1 contains the proofs.

For the relaxed problem to be well-defined, the derivative $\partial_a U(v, a)$ must be well defined.

Remark B.1 in the appendix shows that this is true under our assumptions, and moreover that we can calculate the derivative by differentiating under the integral sign. Henceforth, we will use differentiation under the integral sign without referencing remark B.1.

Throughout this section, fix a regular intended action a_0. To simplify notation, we omit the dependence on a_0 whenever it is clear, writing for example $U(v)$ instead of $U(v, a_0)$.

2.5.1. Solution to the Relaxed Problem

We first show that the relaxed cost minimization problem has an almost everywhere unique solution, and that this solution has a simple formula in terms of Lagrange multipliers. The form of the solution is based on the first order condition, and is similar to standard formulas in the literature, such as in the case without limited liability.[7]

The relaxed cost minimization problem is convex. Define its Lagrangian as

$$\mathcal{L}(v, \lambda, \mu) := W(v) + \lambda \left(\bar{U} - U(v)\right) + \mu(-U_a(v)). \tag{2.2}$$

Heuristically differentiating this Lagrangian with respect to $v(y)$ and setting the derivative to zero gives

$$k'(v(y))f(y|a) = \lambda f(y|a) + \mu f_a(y|a_0).$$

Dividing by $f(y|a)$ gives

$$k'(v(y)) = \lambda + \mu \frac{f_a(y|a_0)}{f(y|a_0)}. \tag{FOC}$$

Equation (FOC) is the key step in the standard first-order approach literature. Appendix B.1 formally analyzes the convex program, demonstrates existence and uniqueness of the solution, and characterizes the optimal contract based on equation (FOC).

[7]This type of formula for the optimal contract is well-known and central in the literature that assumes the first-order approach. The seminal reference is Holmström (1978), and recent examples with limited liability include Jewitt et al. (2008) and Chaigneau et al. (2022). The survey Georgiadis (2022) has an excellent explanation of the formula in its first paragraphs.

In our case of limited liability, the solution is described more clearly with the following notation. Define the **optimal expected wage** $\omega(\bar{U})$ as the value of the infimum in the relaxed cost minimization problem. Define the **link function** $g : \mathbb{R} \to \mathbb{R}$ as[8]

$$g(z) := k'^{-1}\left(\max\left\{\frac{1}{u'(0)}, z\right\}\right).$$

Define the **score** function[9] as in the statistics literature:

$$S(y|a) := \frac{\partial}{\partial a}\log f(y|a) = \frac{f_a(y|a)}{f(y|a)}.$$

Definition 2.2. *A canonical contract $V(y|\lambda,\mu)$ is defined for λ and μ in $\mathbb{R}$ as*

$$V(y|\lambda,\mu) := g(\lambda + \mu S(y|a_0)). \tag{2.3}$$

The following proposition shows that the relaxed problem typically has a solution and characterizes the solution and Pareto frontier.

Proposition 2.1. *[Solution of the Relaxed Cost-Minimization Problem] There exists $\bar{U}_L$ in $\mathbb{R}$ and $\bar{U}_R$ in $\mathbb{R} \cup \{\infty\}$ such that:*

1. *For $\bar{U} \geq \bar{U}_R$, the relaxed problem is not feasible.*
2. *For $\bar{U} < \bar{U}_R$, the relaxed problem has an almost everywhere unique solution $v^*(y|\bar{U})$. There exist Lagrange multipliers $\lambda^*(\bar{U}) \geq 0$ and $\mu^*(\bar{U}) > 0$ such that the solution is almost everywhere equal to the canonical contract*

$$v^*(y|\bar{U}) = V\left(y|\lambda^*(\bar{U}), \mu^*(\bar{U})\right).$$

[8]The link function's input, z, is a marginal dollar cost of providing one util to the agent (measured in units of $\frac{\$ \text{ of output}}{\text{utility}}$). The link function evaluated at z, $g(z)$, returns the utility level where z is the marginal cost of utility to the agent. Any z below $\frac{1}{u'(0)}$, which is the cheapest possible marginal cost, returns $g(z) = u(0)$.

[9]The score is known as the likelihood ratio in the economics literature. We favor the term score to be in line with the modern literature outside of economics.

3. *For $\bar{U} \leq \bar{U}_L$, we have $\lambda^*(\bar{U}) = 0$. The relaxed optimal contract $v^*(\cdot, \bar{U})$ and optimal expected wage $\omega(\bar{U})$ do not vary with $\bar{U}$ in this range.*

4. *For $\bar{U}_L < \bar{U} < \bar{U}_R$, $\lambda^*(\bar{U})$ is strictly increasing and $\lim_{\bar{U}\to\infty} \lambda^*(\bar{U}) = \infty$. The optimal expected wage $\omega(\bar{U})$ is strictly increasing and strictly convex.*

The Gaussian-log utility example illustrates proposition 2.1. The score is $(y - a)/\sigma^2$, the compensation cost function is $k(v) = \exp(v) - w_0$, and the link function is $\log \max\{z, w_0\}$. This implies the wage function

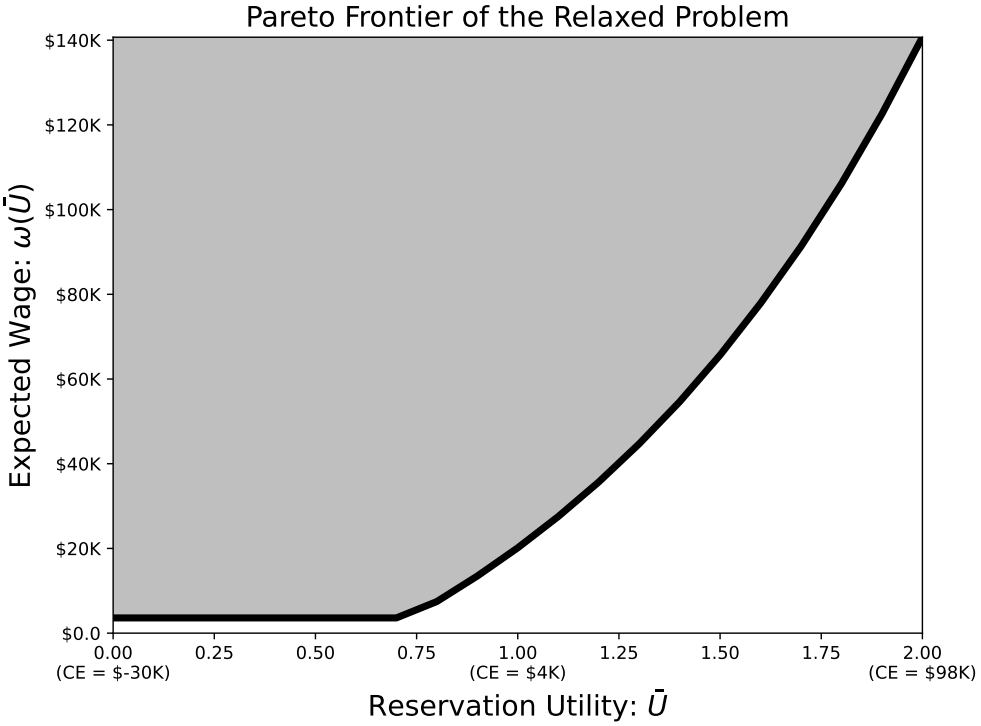
$$k(v^*(y|\bar{U})) = \left[\lambda^*(\bar{U}) + \mu^*(\bar{U})\frac{y-a}{\sigma^2} - w_0\right]^+.$$

This is the piecewise linear solution that we saw numerically in figure 2.1a. The convex Pareto frontier is illustrated in figure 2.4. In the log utility case, $\bar{U}_R = \infty$, as the principal can give arbitrarily high utility to the agent.

The formula in proposition 2.3 is essentially the standard first-order approach's equation (FOC), but properly accounting for limited liability. The intuition is the following. The principal would like to always pay the agent $g(\lambda^*(\bar{U}))$, which is the utility level where the marginal cost of providing utility to the agent is $\lambda^*(\bar{U})$. However, incentive compatibility requires that payment depends on whether there is statistical evidence of high effort. The contract, therefore, pays more if the score is positive and less if the score is negative.

Proposition 2.1 deviates substantially from the literature without limited liability. Our formula for the optimal contract is different because it gives the agent a constant payment of 0 for sufficiently low outcomes, where the limited liability constraint binds (see the kink in the wage function in Figure (2.5a). Proposition 2.1 guarantees existence of an optimum in the relaxed problem, which is impossible in a model without limited liability, where the relaxed problem often has no solution. This stark difference can be seen in the Gaussian-log utility example. Without limited liability, any contract satisfying the first-order conditions would pay a *globally* linear wage. But this would imply negative consumption in some states, and thus $-\infty$ utility to the agent. Indeed, the Gaussian-

Figure 2.4: Pareto Frontier with Gaussian Distribution and Log Utility



Note: This figure displays the relaxed problem's Pareto frontier. The agent's utility is $u(x) = log(\frac{x}{20K})$, the cost function is $c(a) = \theta a^2$, where $\theta = .01$, and the agent's starting wealth is $w_0 = 50K$. The output is $y \sim N(a, (20K^2))$, where a is the agent's action. The intended action is $a_0 = 100K$.

log utility example illustrates Mirrlees (1999)'s classic point that optimal contracts may not exist without limited liability.

2.5.2. The Case of High Reservation Utility

We now demonstrate that, for sufficiently high reservation utility, the solution to the relaxed cost minimization problem also solves the original cost minimization problem. To do so, we show that the relaxed problem's solution satisfies the global incentive compatibility constraint. This is achieved by demonstrating the stronger result that the agent's utility $U(v^*, a)$ is concave in a at the relaxed optimal contract.

Proposition 2.2. *[Concavity of the Agent's Problem for High Reservation Utility]*

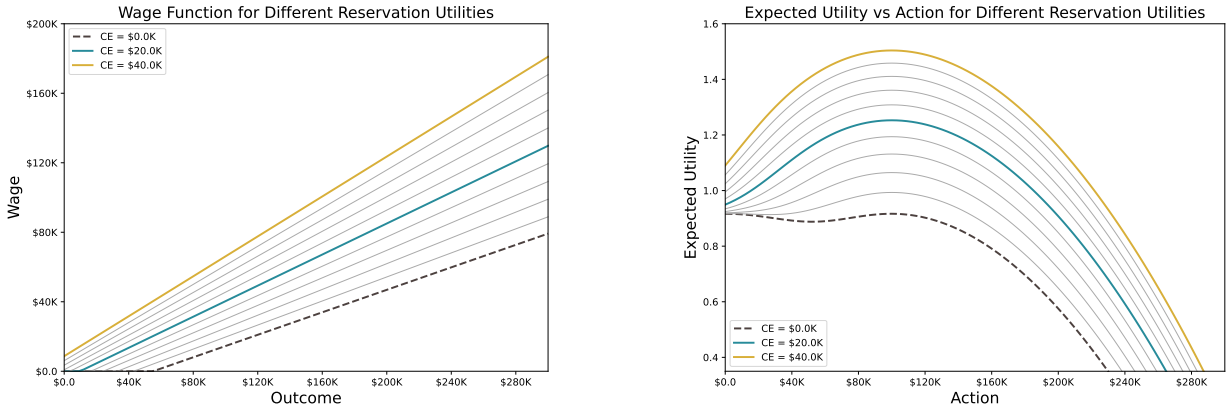
There exists $U^* < \bar{U}_R \in \mathbb{R}$ *such that for all* $\bar{U} > U^*$*, and all* $a \in \mathcal{A}$*,*

$$U_{aa}(v^*(y|\bar{U}), a) \leq 0.$$

The proposition shows that high reservation utility is crucial for the first order approach's validity, and the first order approach may not be valid at low reservation utility. Figure (2.5) helps illustrate this result by plotting the relaxed optimal contract, and the agent's expected utility versus her effort under the relaxed-optimal contract in the Gaussian-log utility example. The charcoal line represents the expected utility function for the relaxed-optimal contract with the lowest reservation utility, $\bar{U} = u(0)$. When reservation utility is low, the agent's problem is not concave, and choosing an action very close to 0 is a profitable global deviation for the agent. The limited liability constraint makes low effort appealing for the agent; it ensures that the agent can always achieve utility strictly greater than $u(0)$ by exerting no effort. If the IR constraint binds, as it does in many of our examples, then the agent's expected utility from choosing the intended action is $\bar{U}$, and the first order approach surely fails when $\bar{U} = u(0)$.

Figure (2.5) also illustrates how, as reservation utility increases, the first-order approach becomes valid. The IR constraint ensures that $U(v^*, a_0)$ must be greater than $\bar{U}$. The rightmost local

Figure 2.5: Relaxed Optimal Contracts with Gaussian Distribution and log Utility



(a) Canonical Contracts

(b) Expected Utility

Note: Panel a displays the wage function that solves the relaxed problem at different values of $\bar{U}$, and Panel b displays the agent's expected utility from a grid of possible actions given the corresponding wage functions from Panel a. The values of $\bar{U}$ are expressed in terms of certainty equivalents. The agent's utility is $u(x) = log(\frac{x}{20K})$, the cost function is $c(a) = \theta a^2$, where $\theta = .01$, and the agent's starting wealth is $w_0 = 50K$. The output is $y \sim N(a, (20K^2))$, where a is the agent's action. The intended action is $a_0 = 100K$.

maximum, $U(v^*, a_0)$, therefore, increases as $\bar{U}$ increases. The leftmost local maximum also rises, but much more slowly. In the example, the first-order approach becomes valid with a very small increase in $\bar{U}$ above $u(0)$, as the rightmost local maximum overtakes the leftmost local maximum. Concavity of $U(v^*, a)$ on the other hand, usually requires much larger increases to $\bar{U}$, so, there are many values of $\bar{U}$ where the first-order approach is valid, even though $U(v^*, a)$ is not concave in a. Thus, concavity of $U(v, a)$ is a sufficient, but not necessary condition for the first-order approach's validity.

We now explain why the agent's problem is concave for high reservation utility, but not for low reservation utility. The relaxed optimal contract has two distinct regions. When the outcome, y, is below a threshold, the limited liability constraint binds, and the contract specifies a constant wage of 0. When y is above the threshold, the agent's payment is dictated by equation FOC. Figure 2.5 illustrates how the threshold depends on the agent's reservation utility. As reservation utility increases, the threshold shifts to the left.

The region where the limited liability constraint binds introduces convexity to the agent's

problem. When the limited liability constraint binds, the agent is paid 0, and marginal increases to the agent's effort do not change her payment. The agent's returns to effort are therefore proportional to the probability the agent receives a positive payment. This probability is increasing on her effort, so the agent's expected wage is a convex function of the agent's effort.

Consider the Gaussian-log utility example. As we have seen, the wage function is

$$w(y) = \left(\lambda + \mu\frac{y - a_0}{\sigma^2} - w_0\right)^+.$$

The outcome y can be decomposed into a normally distributed shock x plus the action a. Substituting into the wage function yields

$$w(y) = \left(\lambda + \mu\frac{x + a - a_0}{\sigma^2} - w_0\right)^+.$$

The derivative of the expected wage with respect to the action is

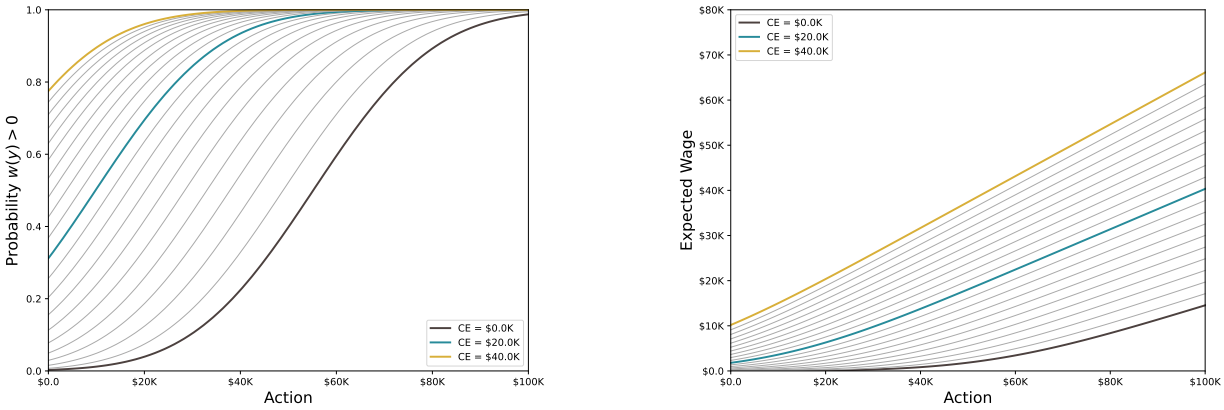
$$\frac{\partial \mathbb{E}[w(y)]}{\partial a} = \frac{\mu}{\sigma^2} \cdot P\left(\lambda + \mu\frac{x + a - a_0}{\sigma^2} - w_0 > 0\right).$$

The agent is only exposed to the outcome when x is sufficiently large. If x is too small for the agent to get paid, the agent receives no benefit from marginally increasing effort; she is paid 0 regardless. The agent experiences marginally increasing returns to effort because the probability that the agent's effort affects her payment increases on a.

As reservation utility increases, the threshold for positive payment shifts to the left, so the agent's probability of payment increases. Figure 2.6a illustrates that when reservation utility is high, the probability of payment is close to 1 even for low effort levels. Lemma B.7 formalizes the result that the probability of payment converges to 1 as reservation utility converges to $\bar{U}_R$. When reservation utility and the probability of payment are low, the probability of payment increases rapidly on effort, and the agent's expected wage function is highly convex (see Figure 2.6b). At high reservation utility, payment is likely regardless of the effort level, so the agent's expected wage

is nearly linear. The agent's utility function is concave, and the cost function is convex, so the agent's problem is globally concave as long as the limited liability constraint does not distort the wage function too much.

Figure 2.6: Probability of Payment and Expected Wages in Gaussian-Log Utility Example



(a) Probability of Payment

(b) Expected Wage

Note: Panel A plots the probability that the agent receives a strictly positive wage against the agent's action, under the relaxed optimal contract for different values of reservation utility. Panel B plots the expected wage against effort for the same contracts. The parameters of the problem are as in Figure 2.5.

The proof of lemma B.7 illustrates why the probability of payment converges to 1 as reservation utility gets large. When reservation utility is large, the agent receives a large payment in nearly all states of the world where she is paid. If she were likely to be paid zero, there would be an incentive to work harder than the intended effort level to ensure that she receives the large payment. Therefore, for the first order condition to hold, the probability that the agent receives no payment must converge to 0.

Once this is established, proposition 2.2 follows naturally. Under our assumptions, the optimal contract v(y) is concave in the region where $w(y) > 0$. Thus, since the option-like region of the contract is far to the left, $U(v^*, a)$ is concave in a. Our result is closely related to Jewitt (1988)'s classic result. He shows that the agent's problem is concave in a similar setting, but without limited liability. In his setting, optimal contracts always have the concave shape that our contracts exhibit in the region $w(y) > 0$. Thus, Jewitt (1988) shows that the agent's problem is concave, and this

holds for any reservation utility. Unfortunately, without limited liability there are often no optimal contracts, as we already demonstrated in the Gaussian-log utility example.

2.5.3. Proof of Theorem 2.1

Theorem 2.1 follows trivially from propositions 2.1 and 2.2.

Proof of Theorem 2.1. Parts (1) and (2) of Theorem 2.1 are included in Proposition 2.1. Part (3) follows from Proposition 2.2. For any $\bar{U} \geq U^*$ we have that $v^*(\bar{U})$ is concave, and thus is a solution to the cost minimization problem. Any other solution also solves the relaxed cost minimization problem, so equals $v^*(\bar{U})$ almost everywhere. □

2.6. Applications and Extensions

2.6.1. Closed Form Solutions

Theorem 2.1 implies that optimal contracts have simple functional forms in parametric settings. Here we describe the solutions for common parametrization and note their properties.

Optimal contracts depend on effort cost, risk preferences (which determine the link function g), and on the distribution of output (which determines the score S). Tables 2.1 and 2.2 provide basic formulae for the link and score functions that make up optimal contracts $g(\mu + \lambda S(y|a_0))$.

Table 2.1: Utility Functions, Link Functions, and Wage Functions

	Utility Function $u(x)$	Link Function $g(z)$	Wage Function $w(z)$
Log	$\log(x + w_0)$	$\log(\max(w_0, z))$	$(z - w_0)^+$
CRRA	$\frac{(x+w_0)^{1-\gamma}}{1-\gamma}$	$\frac{\max(w_0^\gamma, z)^{\frac{1-\gamma}{\gamma}}}{1-\gamma}$	$\left((z^+)^{\frac{1}{\gamma}} - w_0\right)^+$
CARA	$\frac{-\exp(-\alpha(x+w_0))}{\alpha}$	$-\frac{1}{\alpha \max(\exp(\alpha w_0), z)}$	$\frac{(\log^+ z - \alpha w_0)^+}{\alpha}$

Note: The utility function is the agent's utility from consumption given starting wealth w_0 and a transfer, x. The link and wage functions are in terms of z, which is a function of the outcome, y: $z(y) = \lambda + \mu S(y|a_0)$.

Linearity Linear contracts play a prominent role in contract theory. It has long been noted that linear contracts are common, but that they only arise under relatively special assumptions

Table 2.2: Error Distributions

Distribution	**Probability Density**	**Score Function** $S(y\|a)$	**Mean**
Gaussian	$\mathcal{N}(a, \sigma^2)$	$\frac{y-a}{\sigma^2}$	a
Log Normal	$\frac{1}{y\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(\log(y)-a)^2}{2\sigma^2}\Big)$	$\frac{\log(y)-a}{\sigma^2}$	$\exp\Big(a+\frac{\sigma^2}{2}\Big)$
Poisson	$\frac{a^y e^{-a}}{y!}$	$\frac{y-a}{a}$	a
Exponential	$\frac{1}{a} e^{-\frac{y}{a}}$	$\frac{y-a}{a^2}$	a
Bernoulli	$a^y(1-a)^{1-y}$, $y \in \{0,1\}$	$\frac{y-a}{a-a^2}$	a
Geometric	$\Big(1-\frac{1}{a}\Big)^{y-1}\Big(\frac{1}{a}\Big)$, $y \in \{1,2,\dots\}$	$\frac{y-a}{a^2-a}$	a
Binomial	$\binom{n}{y} a^y (1-a)^{n-y}$, $y \in \{0,\dots,n\}$	$\frac{y-na}{a-a^2}$	na
Gamma	$f(y \mid n, a) = \frac{y^{n-1} e^{-\frac{y}{a}}}{\Gamma(n)\, a^n}$	$\frac{y-na}{a^2}$	na
Student's t	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\,\sigma}\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)^{-\frac{\nu+1}{2}}$	$\frac{(\nu+1)(y-a)}{\nu\,\sigma^2+(y-a)^2}$	a
Exponential Family	$h(y)\, \exp\Big(\eta(a)\, T(y) - A(a)\Big)$	$T(y)\, \frac{d\eta(a)}{da} \;-\; \frac{dA(a)}{da}$	*(Not specified)*
$y = a + X, X \sim h$	$g(y-a)$	$-\frac{g'(y-a)}{h(y-a)}$	$a + \mathbb{E}[X]$
$y = aX, X \sim h$	$\big\lvert\frac{1}{a}\big\rvert\, h\Big(\frac{y}{a}\Big)$	$-\frac{1}{a} - \frac{y}{a^2}\frac{g'(\frac{y}{a})}{g(\frac{y}{a})}$	$a\,\mathbb{E}[X]$

Note: This table presents probability the PDF, score function, and means of probability distributions as functions of the agent's chosen action, a.

(Holmstrom and Milgrom, 1987; Carroll, 2015).

Our model yields piecewise linear contracts in a number of examples, as seen in the tables and in the Gaussian-log utility example. The key ingredients are log utility, which makes the wage function linear in the score, and a linear score function. This includes many distributions, because the score is linear for distributions in an exponential family with linear sufficient statistics. That is, when $f(y, a)$ is of the form

$$f(y|a) = \exp\left(\eta(a)y + A(a)\right). \tag{2.4}$$

We note this as follows:

Remark 2.1 (Linear Contracts)**.** *Assume that utility is log ($u(x) = \log(w_0 + x)$), and that the distribution of outcomes is in an exponential family with linear sufficient statistic (as in equation 2.4). Then the relaxed optimal contract wage function is piecewise linear. This includes the Gaussian,*

exponential, Poisson, and gamma distributions.[10]

Concave and Convex Contracts The optimal contract is always at least partially convex because the limited liability constraint requires that $w(y) = 0$ for all y less than some threshold, $\underline{y}$. The region where the limited liability constraint does not bind, however, can be convex or concave depending on the agent's risk aversion.

Remark 2.2. *Suppose the score function is linear, and let $\mu S(y|a_0) = ky$. Then, if the agent has CRRA utility, the region of the optimal contract where the limited liability constraint does not bind is convex if the risk aversion parameter $\gamma > 1$ and concave if $\gamma < 1$:*[11]

$$w(y) = (\lambda + ky)^{\frac{1}{\gamma}} \text{ for } y > \underline{y}.$$

If the agent has CARA utility, the same region of the optimal contract is logarithmic:

$$w(y) = \frac{\log(\lambda + ky)}{\alpha} - w_0 \text{ for } y > \underline{y}.$$

The remark shows that the shape of the optimal contract depends heavily on the agent's utility function, and a wide variety of contracts are potentially compatible with the theory. Our result is contrary to Conlon (2009)'s view that convex contracts, like CEO compensation with stock options, are incompatible with the first-order approach. In fact, the optimal contract with CRRA utility and $\gamma < 1$ closely approximates a CEO compensation package of stock options with many strike prices. To see this, consider a CEO who receives n_i options with strike price s_i that expire at the

[10] Note that the remark covers distributions with support different than $\mathbb{R}$, such as the exponential. These distributions do not satisfy Assumption (2.3). Nevertheless, the proof of proposition 2.1 does not use that the support is $\mathbb{R}$, so the remark holds. However, our proof that the relaxed optimal contract is optimal does use that the support is $\mathbb{R}$. Therefore, we do not know whether Theorem 1 can be extended to different supports, and thus whether the remark can be extended to optimal contracts in the case of sufficiently high reservation utility. We conjecture based on numerical results that this is true and hope to include these facts in our next revision.

[11] Assumption 2.4 requires that $\gamma > \frac{1}{2}$.

end of the year. The CEO's wage is

$$w(y) = \sum_i n_i (y - strike_i)^+,$$

where y is the stock's end of year price price. The wage function's slope at y is $\sum_{strike_i < y} n_i$; the wage is a piecewise-linear function where the slope increases as y increases. The options package wage may be a discretization of the contract the model predicts for an agent with CRRA utility with $\gamma < 1$, and a linear score function. Its slope is also increasing on the outcome and equals $\frac{k}{\gamma}(\lambda + ky)^{\frac{1-\gamma}{\gamma}}$ for some constant k.

Remark 2.2 affirms Holmstrom and Milgrom's (1987) view that the principle agent model does not robustly predict linear wages. The optimal wage is only linear if the agent's risk aversion parameter is precisely $\gamma = 1$. However, if the agent's risk aversion is approximately one, the optimal contract will be approximately linear, and using a linear contract may be practical.

Limitations and numerical methods. The formulae have two main limitations. First, there is no general closed form solution for the Lagrange multipliers λ and μ. Thus, Theorem 2.1 guarantees that optimal contracts have the described functional form, but the Lagrange multipliers have to be calculated numerically. Second, as noted in Theorem 2.1, these simple formulae do not hold for sufficiently low reservation utility.

To address these limitations, we provide accompanying code to numerically solve for optimal contracts. The numerical methods solve both the relaxed problem, and also the full cost minimization problem, in the case where only a finite number of global constraints bind. This is the case in all the experiments that we conducted. Both are convex optimization problems, and can be solved at trivial computational cost in all our experiments. The code implements all the common specifications in tables 2.1 and 2.2.

2.7. Conclusion

This paper shows that when an agent's reservation utility is sufficiently high, the longstanding concerns about the validity of the first-order approach largely disappear. Despite the possibility of

multiple local maxima under low reservation utility, forcing the principal to provide higher overall utility eliminates global deviations, ensuring both existence of an optimal contract and applicability of the standard calculus-based approach. Our results hold under natural assumptions and extend to many commonly used distributions and preferences. This suggests that the first-order approach is especially relevant in competitive labor markets or other environments where agents have strong outside options.

In particular, our results show that the many existing results in the literature that have been proven assuming the first-order approach hold in this broader array of settings, include many settings with option-like contracts, and even contracts with wages that are convex functions of output. We hope that this result and the basic calculus formulae we provide will be useful in applied and theoretical work.

CHAPTER 3

HETEROGENEOUS VALUE-ADDED FOR PRIORITY DESIGN IN NEW YORK CITY HIGH SCHOOLS

3.1. Introduction

This paper investigates a simple and understudied question: which students should attend which schools? Different types of students may experience heterogeneous benefits from attending different types of schools. If so, matching students to the schools that best suit their needs could substantially improve school systems' overall quality. I study heterogeneous value-added in New York City public schools, and find that stronger students benefit the most from attending academically competitive schools, while weaker students tend to make less progress than they would have made at less competitive schools. Students with disabilities and English language learners, in particular, do worse in the most academically rigorous settings. The substantial heterogeneity that I document highlights the potential for better student-school matching to generate large gains in educational achievement.

In most school districts that use centralized matching, schools give priority to students based on students' characteristics, such as geographical location, whether the student has a sibling who attended the school, where they attended a lower school, socioeconomic class, or academic achievement. School districts can design admissions priorities to maximize the efficiency of the match by giving priority to students who would benefit the most from attending. Designing these priorities is often a contentious public policy question. In 2014, Boston Public Schools reformed its school assignment system to increase geographic access and reduce transportation burdens by implementing a home-based plan that gave students priority at nearby schools. A recent policy paper for the San Francisco Unified School District optimizes neighborhood priority design to reduce segregation (Allman et al., 2022). Rigorous measures of heterogeneous value-added can help inform these debates and ensure that students have access to schools where they will succeed.

This paper focuses on New York City's top quality high schools, which I define as the top 10%

of schools by matriculants' average middle school test scores. Students with higher scores on middle school standardized tests benefit more from attending these schools than other students. For every standard deviation increase in middle school standardized test scores, I find that attending a top quality school increases students' PSAT scores by about $.05-.15$ standard deviations, depending on the specification. Many of these top quality schools are a special category of school called "screened schools", which use prior academic achievement to rank students. The priority structure at screened schools is debated and adjusted on a near yearly basis, and my results are directly applicable to that debate.

In 2021, the de Blasio administration implemented a centralized priority structure for most screened schools, although a small subset was still permitted to choose their own priorities. The priority structure eliminated standardized testing, both because the COVID pandemic interrupted standardized testing, and because the new administration thought tests were inequitable and poor predictors of future performance. Rather than strictly ranking students based on their grades, the new priority structure gave preference to students with middle school GPAs above 85% (roughly 60% of students in the city), and used a lottery to tiebreak between students who met the GPA qualification.

My results suggest that the de Blasio administration's policies reduced aggregate value-added. I find that students with higher middle school standardized test scores have substantially higher value-added from attending screened schools, so standardized test scores should be used in the priority structure. Furthermore, I find that the value-added of attending top schools is increasing on previous academic achievement, and a strict, or at least less coarse priority structure, would allow more top students to attend highly competitive schools. Mayor Eric Adams' administration did tighten the requirements for top academic priority at screened schools by limiting the top priority group to students with a GPA of at least 90% whose GPA also ranked in the top 15% of their middle school or the city.

Many New York City schools give priority to students with disabilities or students from disadvantaged backgrounds. In an effort to desegregate schools, New York City piloted its diversity in

admissions program in 2015 (Wall, 2015). The program gives schools the option to reserve some seats for students who are eligible for free or reduced price lunch, or who are English language learners. The admission threshold for the reserved seats may be lower than the admission threshold for the open seats, increasing the number of students who are eligible for reserved seats who are admitted. In the 2024 admission cycle, 43 high schools participated in diversity in admissions, including Beacon High School and Bard High School, two of the most competitive screened schools. Similarly, many schools reserve seats for students with disabilities. I find that students with disabilities and students from disadvantaged backgrounds perform better when they do not attend elite high schools, indicating that some of these schools should not reserve seats for students from these backgrounds.

This paper contributes to two literatures: the school value-added literature and the priority-design literature. Like many previous papers, I use randomization in school assignment arising from admissions lotteries to estimate school value-added. There is a substantial literature using lotteries to study the impact of charter schools and other alternatives to traditional public education, including Hoxby and Rockoff (2004), Abdulkadiroglu et al. (2011a), Dobbie and Fryer (2011), Bloom and Unterman (2014) and Abdulkadiroglu et al. (2013). Unlike these earlier studies, my empirical method relies on recent advances in fully utilizing randomization from lotteries in school choice mechanisms. Whereas previous studies only used lotteries at oversubscribed schools, Abdulkadiroglu et al. (2017) pioneers a new approach which allows researchers to take advantage of all the randomization inherent in centralized school choice mechanisms where schools with coarse preferences use lotteries to break ties. Each student has a probability of assignment at each school based on the students' and schools' preferences. Conditional on this probability, which is analogous to a propensity score, school assignment is random, which allows researchers to identify the causal impact of attendance. Using this methodology, I identify heterogeneous value-added to determine which types of students benefit the most from attending which types of schools.

This paper joins the growing priority design literature. Early papers in the literature focused on how to tiebreak coarse priority structures. Abdulkadiroglu et al. (2009) showed that using a

single lottery tiebreaker at all schools increases the number of students who receive their top choice, compared to using a unique lottery tiebreaker, and Jeong (2022) finds that multiple tiebreakers improve diversity compared to single tiebreakers. Clark and Coate (2023) shows that in a theoretical setting where students value their peers' socioeconomic status, giving priority to disadvantaged students may increase the efficiency of the match. Shi (2022) provides a simple linear programming algorithm to compute optimal priorities given the market designer's objective function. This paper's empirical estimates of value-added could be used as an input to Shi (2022)'s algorithm to compute the optimal priority structure for a designer whose objective is to maximize value-added.

3.2. Data

I received data from the New York City department of education on all students who participated in the high school admissions process in the fall of 2021 and 2022. The dataset includes the students' and schools' preferences submitted to the centralized match algorithm, students' middle school and high school standardized test scores and grades, and demographic variables. Table 3.1 presents basic summary statistics. This paper studies the heterogeneous value-added of different types of schools, so the summary statistics table separates students into groups by the type of high school they enrolled in during 9th grade. I consider 4 different types of high schools: screened, top quality, top quality screened, and other screened. Screened schools are schools that use academic criteria to determine admissions priority, and top quality schools are schools whose matriculating students' average middle school standardized test scores are in the top 10% of all schools in New York City. Top quality screened schools are schools that are both screened and top quality, and the remaining screened schools are called other screened schools.

The middle school tests variable is the normalized average of students' normalized middle school standardized test scores. Students in top quality schools and top quality screened schools score .78 and .8 standard deviations above average respectively. PSAT score is a key outcome variable I use to measure students' high school academic achievement. The exam is typically taken at the end of 10th grade or the beginning of 11th grade in preparation for the SAT college admissions test. Students in top quality schools and top quality screened schools perform nearly a standard deviation

Table 3.1: Data and Summary Statistics

	All	Screened	Top Quality	Top Quality Screened	Other Screened	Portion Valid
Middle School Tests	0.00	0.21	0.78	0.80	0.01	0.84
	(1.00)	(1.01)	(0.84)	(0.84)	(0.98)	
Math PSAT	0.00	0.23	0.73	0.78	0.01	0.62
	(1.00)	(1.03)	(0.98)	(0.98)	(0.96)	
English PSAT	0.00	0.21	0.72	0.74	0.00	0.62
	(1.00)	(1.01)	(0.95)	(0.95)	(0.95)	
Hispanic	0.42	0.36	0.21	0.20	0.41	0.94
Black	0.23	0.18	0.15	0.15	0.19	0.94
Asian	0.18	0.25	0.33	0.36	0.21	0.94
White	0.15	0.19	0.27	0.26	0.17	0.94
English Language Learner	0.12	0.12	0.04	0.04	0.15	0.95
Disability	0.21	0.19	0.16	0.16	0.20	0.95
Poverty	0.74	0.70	0.56	0.58	0.74	0.95
Male	0.51	0.50	0.46	0.48	0.51	0.95
Sample Size	145279	74532	21127	19012	55520	NA

This table presents summary statistics of students, with separate columns for students who enrolled in each of the types of schools studied in this paper. Screened schools use academic screening for admissions, top quality schools are the top 10% of schools in the city by matriculating student's average middle school test score. Top quality screened schools are both screened and top quality, and other screened schools are the remaining screened schools. Middle school tests is the normalized average of normalized middle school standardized test scores. The PSAT scores variables are normalized. The portion valid column is the fraction of observations for which I have valid observations.

better than average on the PSAT. Students in the remaining screened schools, on the other hand, have average PSAT scores.

The students in the data are majority Black or Hispanic, and nearly three fourths of students are in poverty. However, only 35% of students in top quality screened schools are Black or Hispanic, and only 58% are in poverty. The fraction of students who are English language learners, or who have disabilities is also substantially lower at the top quality screened schools than in the city's school system as a whole.

The portion valid column is the portion of the observations in the dataset for which I have valid data. I have middle school test scores for 84% of observations. The missing observations are predominantly comprised of students who did not attend middle school in New York City. The data contains PSAT scores for 62% of all students, and 74% of students who attended high school in NYC. The test is not mandatory, and many students choose not to take it. PSAT score is a key outcome variable, so non-random missing data is an important identification challenge. Fortunately, at some schools, nearly all students take the PSAT, so I address the selection issue by running a specification

limited to schools where at least 80% of students took the PSAT. Demographic variables are missing a small portion of the time, usually when the student participated in the admissions process, but never enrolled in a New York City middle school or high school.

3.3. Identification

This paper uses Abdulkadiroglu et al. (2017)'s approach to utilize the randomization inherent in centralized school choice mechanisms where schools with coarse preferences use lotteries to break ties. In New York City, each school has criteria they use to sort students into priority groups. Depending on the school, priority groups may depend on geographic location, whether a sibling attends the school, or which middle school the student attended. Screened schools use prior academic achievement to determine priority group, and schools participating in NYC's diversity in admissions program give top priority at a subset of their seats to economically disadvantaged students, or English language learners. Every student is given a single lottery number which schools use to break ties between students in the same priority group. Although the lottery numbers are actually hexadecimal strings, it is without loss think of them as being uniformly distributed between 0 and 1, and assume that higher lottery numbers are preferred.

Students that are in marginal priority groups are randomly assigned to schools because their assignment depends on their lottery number. The mechanism also creates some randomization among students who have a sufficiently high priority group to guarantee admission to a particular school because in order to be admitted, the student must first be rejected at all the schools they ranked higher, which depends on their lottery number. I make use of all of the randomization in the mechanism by computing students' probability of admission to different types of schools, and then control for these probabilities of admissions in my regressions. The key identification condition is that, given the probability of admission, school assignment is random. I refer to these probabilities of admission as propensity scores because the empirical methodology is analogous to propensity score matching.

I use Abdulkadiroglu et al. (2017)'s methodology to compute the probabilities of admission. Define a school's threshold, $T_s = (\bar{a}_s, \bar{l}_s)$ as the worst priority group ($\bar{a}_s$) and lottery number ($\bar{l}_s$)

combination that was admitted to the school. Abdulkadiroglu et al. (2017) show that asymptotically in a large market, student $i's$ probability of admission to school s, p_{is}, is a function of the school's threshold, and can be computed as follows.

$$p_{is} = \begin{cases} 0, & \text{if } a_{is} < \bar{a} \\ 1 - \sum_{k \succ_i s} p_{ik}, & \text{if } a_{is} > \bar{a}_s \\ \max\left(0, 1 - \sum_k p_{ik} - \bar{l}_s\right), & \text{if } a_{is} = \bar{a}_s \end{cases}$$

The first case is when the student's priority group is below the threshold for admission to s, so their probability of admission is 0. In the second case, the student's priority group guarantees admission to s. They will be assigned to s as long as they are rejected at all the schools they rank earlier, so their probability of admission is 1 minus the sum of probabilities of admission at earlier ranked schools. The third case is when the student is in the marginal priority group. The student will be admitted if their lottery number is sufficiently high. Rejections at schools the student ranked earlier reveal information about the student's lottery number. In particular, the student's lottery number is at best 1 minus the sum of probabilities of admission at higher ranked schools, unless the student was guaranteed admission to an earlier ranked school, in which case the probability of admission to this school is 0 anyway. The probability the student's lottery number is sufficiently high is, therefore, their maximum lottery number conditional on previous rejections, $1 - \sum_k p_{ik}$, minus the lottery number needed for admission, $\bar{l}_s$.

I use the observed admission results to identify the school's thresholds for admission. I say that a student *applied* to school s if they include s in their rank order list, and all the schools the student ranked ahead of s rejected them. If the threshold were perfect, then all students who apply to a school with credentials above the threshold would be admitted, and all students who apply to a school with credentials below the threshold would be rejected. Unfortunately, there is some noise in the data, and students who appear to have worse credentials are occasionally admitted above students with better credentials. I use an algorithm that identifies likely thresholds, and then

selects the threshold which minimizes the sum of rejected applicants with scores above the threshold and accepted applicants with scores below the threshold. Table 3.2 shows the threshold's confusion matrix, and demonstrates that the thresholds correctly predict the admission decision for the vast majority of applications.

Table 3.2: Threshold Confusion Matrix

	Predict Reject	**Predict Accept**
Reject	262143	821
Accept	1510	133956

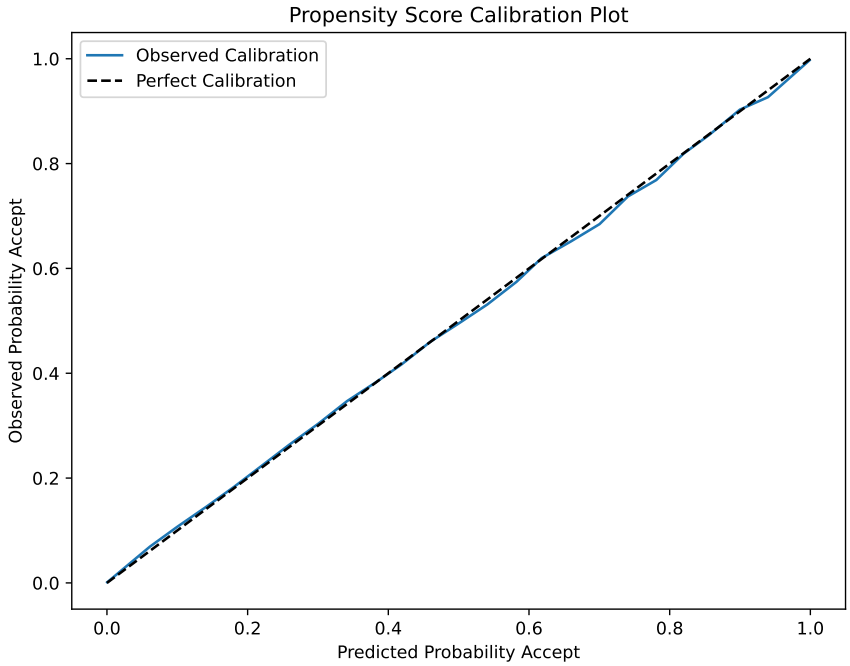
This table compares predicted admission decisions using the thresholds identified by the algorithm to the actual admission decisions observed in the data.

Despite minor noise in the thresholds, the propensity scores are well calibrated, meaning that the students with admission probability x are actually admitted with probability of approximately x. The propensity score calibration plot, figure 3.1, places applications into 25 bins depending on their propensity scores. It plots the portion of the applications in the bin that were actually admitted against the average propensity score of the applicants in the bin. As expected, the observed probabilities of admission precisely match the predicted probabilities of admission.

This paper studies heterogeneous value-added of different types of schools, rather than particular schools. I aggregate the propensity score for admission to particular schools to compute the student's propensity score for a type of school. The student's propensity score for a particular type of school is the sum of the student's propensity score for all schools of that type. I consider 4 different types of schools: top quality schools, screened schools, top quality screened schools, and other screened schools. Top quality schools are schools that are in the top ten percent of all schools by the average middle school standardized test scores of their matriculating class, and screened schools are schools that use academic screening criteria to determine admission. Figure 3.2 shows that the aggregated propensity scores for the different types of schools are well calibrated.

The key identification requirement is that conditional on students' propensity scores, admission is random, and determined solely by the lottery. This means that controlling for probabilities of

Figure 3.1: Individual School Level Propensity Score Calibration



Applications are placed in 25 bins based on their predicted probability of acceptance. Bins start at 0 and each bin is 4 percentage points wide. The x axis is the average predicted probability of acceptance among applications in the bin, and the y axis is the observed fraction of the applications that were admitted.

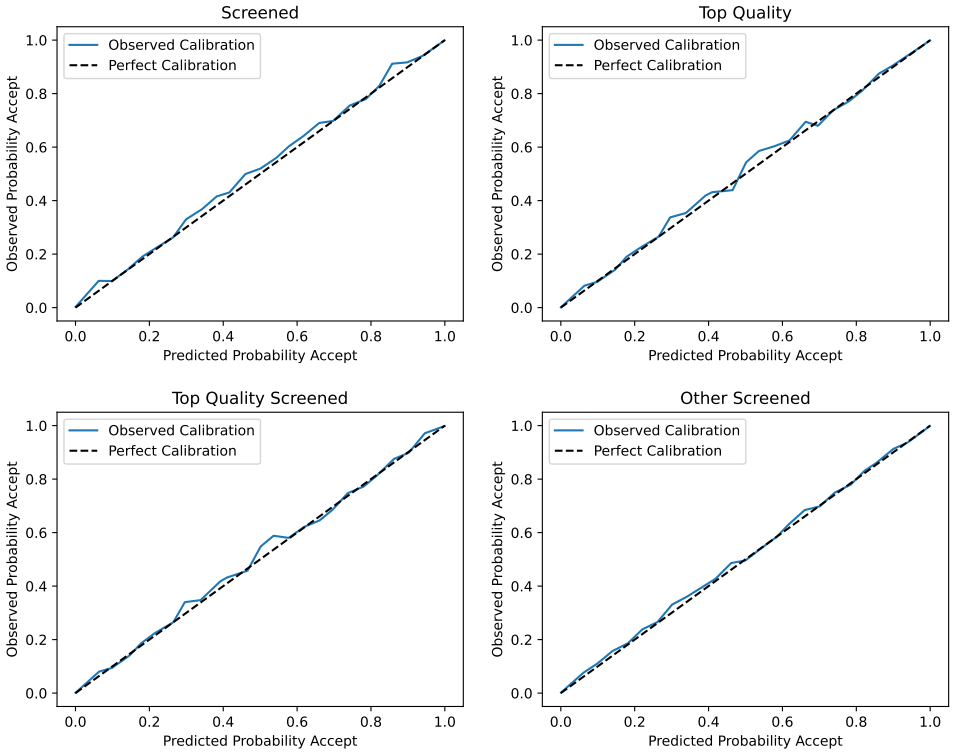
admission, there should be no correlation between student's observable characteristics and whether they are admitted. The balance test regression equation is

$$x_i = \beta_0 + \beta_1 p_i + \beta_2 s_i + \epsilon_i$$

where x_i is an observable characteristic, p_i is a matrix of indicators for propensity scores, and s_i is an indicator for whether the student is admitted to a school of category s.

Table 3.3 presents the coefficients β_2 for a grid of x_i and s_i. Of the 36 coefficients, 4 are statistically significant at the 5% level, and 2 more are statistically significant at the 1% level. This suggests that the computed probabilities of admission are not perfectly accurate, and that admission is not random even after conditioning on probabilities of admission. The threshold

Figure 3.2: School Type Aggregated Propensity Score Calibration



This figure presents aggregated propensity score calibration plots for the four types of schools considered in this study. The bins are the same as in figure 3.1.

confusion matrix, presented above, already indicated there is unexplained noise in some schools' admissions decisions, which likely explains the balance test results. However, the balance test coefficients are all extremely small, which suggests that admission is nearly random conditional on the lottery numbers, and the remaining noise is unlikely to substantially bias the results. I also control for all of these observable characteristics in the main regressions, further reducing any bias resulting from non-random admissions.

3.4. Regression Specifications

Most students attend the school to which they were admitted in the centralized match, but some students do not. Following the centralized match, there is a wait list which gives students the opportunity to pursue a spot at a different school, and students who are not matched to any school are assigned to a school by the city. Even if conditional on the student's propensity score, admission

Table 3.3: Balance Table

	Screened	Top	Top Screened	Other Screened
Middle School Tests	-0.021	0.025	0.023	-0.041***
	(0.011)	(0.015)	(0.015)	(0.012)
Hispanic	0.006	0.008	0.006	0.004
	(0.006)	(0.008)	(0.008)	(0.006)
Black	-0.011**	-0.008	-0.006	-0.004
	(0.005)	(0.007)	(0.007)	(0.005)
Asian	0.006	0.008	0.006	-0.002
	(0.005)	(0.006)	(0.006)	(0.005)
White	0.001	-0.004	-0.004	0.001
	(0.004)	(0.006)	(0.006)	(0.004)
English Learner	0.008**	-0.002	-0.003	0.011***
	(0.004)	(0.005)	(0.005)	(0.004)
Disability	0.010**	-0.002	-0.003	0.012**
	(0.005)	(0.007)	(0.007)	(0.005)
Poverty	-0.002	0.001	-0.001	-0.003
	(0.005)	(0.007)	(0.007)	(0.005)
Male	-0.005	-0.006	-0.008	-0.001
	(0.006)	(0.008)	(0.008)	(0.006)

Coefficients are reported with standard errors in parentheses. Statistical significance at the 10%, 5%, and 1% levels are indicated by *, **, and *** respectively. Middle school tests is the normalized average of the students' normalized middle school standardized test scores. The remaining x variables are all indicator variables. Screened schools are schools that use middle school academic achievement to determine admission priority. Top schools are the top 10% of schools by average standardized test score of students who enroll in 9th grade.

is random, attendance may not be random. I consider two regression approaches-a reduced form approach and an instrumental variable approach-to address this challenge.

The reduced form regression uses admission rather than attendance as the treatment variable. It is an intention to treat analysis, which identifies the value-added of admission to a category of school, without concern for whether the student attended. The regression equation is

$$y_i = \beta_0 + \beta_1 p_i + \beta_2 s_i + \beta_3 x_i + \beta_4 as_i x_i + \epsilon_i,$$

where p_i is a matrix of indicators for propensity scores, and as_i is an indicator for whether the student is admitted to a school of category s, and x_i is a vector of controls. The reduced form approach is simple and does not require the standard instrumental variable assumptions, but likely

attenuates the effect of actually attending a type of school because some students who are admitted do not attend.

The IV regression identifies the value-added of enrolling in different categories of schools, rather than just being admitted. Even conditional on the probability of admission, enrollment is not random, so I need to instrument for enrollment, and any variables interacted with enrollment. Admission is an obvious instrument for enrollment; it is highly correlated with enrollment, and, conditional on propensity scores, it is uncorrelated with unobservable error. In order to identify heterogeneous value-added depending on students' observable characteristics, I require additional instruments for the observables that I want to interact with enrollment. To do so, I regress enrollment on admission to create a predicted enrollment variable. I then use predicted enrollment interacted with the observable characteristic as the instrument for the observable characteristic.

The endogenous regressors are es_i and $es_i x_i$ where es_i is an indicator for enrollment in school category s, and x_i is a vector of observable characteristics. The exogenous regressors are x_i and p_i where p_i is a matrix of indicators for propensity scores. The instrumental variables are the indicator for admission to category s, as_i, and predicted enrollment interacted with observable characteristics, $x_i \hat{es}_i$.

3.5. Results

The regression results show that students with the strongest academic qualifications benefit the most from attending top quality schools, whereas students with weaker academic backgrounds benefit from attending screened schools that are not top quality. Tables 3.5 and 3.6 present reduced form and instrumental variable regression results for the impact of attending top quality schools on PSAT scores. In the full sample (column 1), the reduced form and IV results show that attending a top quality school increases students math PSAT scores by approximately .15 standard deviations for every 1 standard deviation improvement in middle school standardized tests score. To combat potential selection bias, the regressions in column 2 subset to schools where more than 80% of students take the PSAT. The effect is smaller-.062 and .047 standard deviations in the reduced form and IV regressions respectively-in this sample. The difference may be attributable to selection

effect, but could also be because the set of schools being compared is different. The results for the English PSAT score are similar. In the full sample, the benefit of attending a top quality school is .07 standard deviations higher for every standard deviation increase in middle school test performance, but the effect is indistinguishable from 0 in the sample of schools where most students take the PSAT. The regression results for top quality screened schools (see tables 3.7 and 3.8) are qualitatively similar to the results for top quality schools in general, although the effect sizes tend to be smaller.

Students with weaker academic backgrounds flourish at screened schools with average middle school standardized test scores outside the top 10% of all schools in the city. Reduced form and instrumental variable regression results for the impact of attending other screened schools on PSAT scores are in tables 3.9 and 3.10 respectively. The reduced form regressions in the full sample and in the sample of schools where most students take the PSAT show increases on both the math and English sections of the PSAT of about .05 standard deviations for every one standard deviation decrease in student's middle school test scores. The IV regression coefficients are slightly larger. The high value-added for weaker students at screened schools that are not top quality may be because the students benefit from avoiding top quality screened schools. These students perform better in an environment where their previous test scores are similar to their classmates.

My regression results also show that disadvantaged students tend to benefit from attending screened schools that are not top quality, and do worse when they attend top quality schools. The regression results show negative value-added at top quality schools for students with disabilities, students in poverty, English language learners, Hispanic students, and Black students. Attending top quality schools is especially detrimental for English language learner's English PSAT scores, with all of the specifications showing it reduces their scores by approximately .2 standard deviations. Hispanic and Black students have large and positive value-added from attending other screened schools. The reduced form results show small, but positive value-added at other screened schools for students in poverty, whereas the IV results show slightly negative value-added. Both the reduced form and IV results show negative value-added from attending other screened schools

for students with disabilities. Just like at top quality schools, other screened schools have large negative coefficients on English PSAT score for English Language learners.

Many students choose not to take the PSAT, so the results could be biased by selection if there are students who would only take the PSAT if they attended one type of school, but not another. To address this concern, I ran separate regressions for every school type of an indicator for whether the student took the PSAT on the student's observable characteristics, propensity score, and an indicator for whether they enrolled in the given school type. The coefficients estimating the impact of attending a type of school on a student's probability of taking the PSAT are in table 3.4. There is significant selection in the full sample-students who attend top quality schools or top quality screened schools are 4.3 percentage points more likely to take the PSAT, and students who attend other screened schools are 2.3 percentage points less likely to take the PSAT. However, there is little evidence of selection when limiting the sample to schools where at least 80% of students took the PSAT. In this sample, students who attend other screened schools are 1.3 percentage points less likely to take the PSAT, and the impact of attending top quality schools and top quality screened schools is close to 0 and precisely estimated. The results that subset to the sample of schools where most students take the PSAT may be more reliable because there is less selection bias. However, it is also important to note that changing the sample of schools necessarily changes the value-added comparison, so differences between the results may not be attributable to selection bias.

Table 3.4: Selection Test

	Full Data	80%+ PSAT Subset
Screened	-0.001	-0.019***
	(0.006)	(0.006)
Top Quality	0.043***	-0.006
	(0.008)	(0.007)
Top Quality Screened	0.043***	-0.007
	(0.008)	(0.007)
Other Screened	-0.023***	-0.013**
	(0.006)	(0.006)

This table displays the coefficient on attending each type of school from a regression of an indicator for taking the PSAT on observable characteristics and propensity scores. The first column uses the entire sample and the second column subsets to schools where at least 80% of students took the PSAT. Standard errors in parentheses. *** $p < .01$, ** $p < .05$, * $p < .1$.

I consider another outcome variable, students' rank within their class, as an additional robustness check. Class rank is defined as the fraction of students in a student's class who had a lower grade than that student. I have valid grades for nearly every student who attended high-school in NYC, so unlike in the PSAT data, there is little concern about selection bias. Class rank provides a relative, rather than objective measure of academic achievement, so the results are more difficult to interpret. Still, grades are heavily considered in college admissions, so they are an important outcome in their own right. The regression tables are in the Regression Results section. They show that students from disadvantaged backgrounds' class ranks are heavily reduced when they attend the most academically competitive schools.

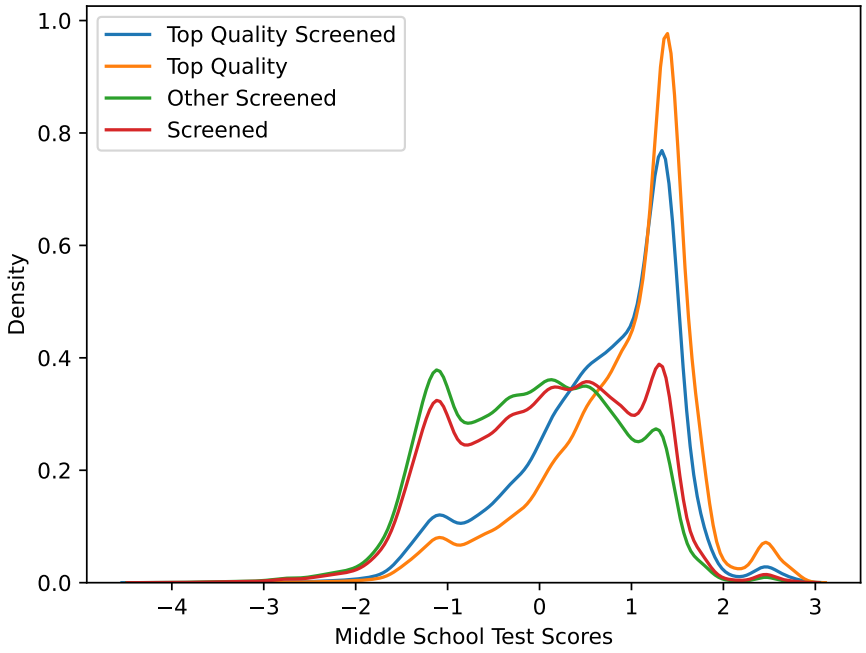
3.6. Discussion

If students who perform better on middle school standardized tests benefit more from attending top quality schools, New York City should consider revising the current priority structure for screened schools, many of which are top quality. The current priority structure ignores standardized tests scores, and only uses middle school grades to determine academic priority. If standardized test scores provide valuable information about which students will succeed at top quality schools, then the schools may want to consider them in their priority structures. In addition to ignoring standardized test scores, the current academic priority structure is coarse. All students whose grades are above a threshold are treated equally, and ties are broken by lottery. If, as my results suggest, the students with the best academic qualifications benefit the most from attending top quality schools, then the city should consider using a strict priority structure, which guarantees that students with the best academic qualifications are admitted.

During the 2021 and 2022 admissions cycles, which are the two years studied in this paper, top academic priority was given to approximately 60% of students in the city. Most screened schools were not permitted to use stricter admissions standards, and could not discriminate between students with top academic priority. This policy resulted in students with a wide range of academic backgrounds attending the top schools. Figure 3.3 plots the density of middle school test scores by school type. By definition, top quality schools contain many more students with elite middle school

test scores. However, these schools also have a substantial mass of students with below average test scores. The regression results show that these students suffer academically when they attend top quality schools, and would perform much better in other screened schools. Similarly, other screened schools have some students with elite middle school test scores. My results suggest substantial gains in value-added from less coarse academic priority structures that would cause some elite students in other screened schools to swap schools with weaker students in top quality schools. In 2023, the city adopted this policy, limiting top academic priority to approximately the top 15% of students.

Figure 3.3: Density of Middle School Test Scores by School Type



This figure displays densities of middle school test scores for students who enrolled in each type of school.

New York City's current priority system gives disadvantaged students preferential treatment at many of the best schools. All screened schools in New York City reserve a portion of their seats for students with disabilities, making it easier for students with disabilities to attend these schools. My results suggest these efforts could be counter-productive, as students with disabilities perform poorly at top quality schools. Many top quality schools participate in NYC's diversity in

admissions program, which gives priority for some seats to English language learners. However, my results show that top quality schools are especially ill-suited to teach English language learners. Schools that adopt priority structures that specifically target disadvantaged students should make sure they have the resources and programming to help those students thrive. Ideally, the schools that are best equipped to teach disadvantaged students should adopt priority structures that make it easier for disadvantaged students to attend.

This paper shows how priority design can improve match efficiency. My results show that different types of students have significant heterogeneous value-added from attending different types of schools. During the years studied, the coarse priority structure at screened schools allowed many students with weaker academic backgrounds to attend elite schools, where they performed poorly. As policymakers continue to revise New York City's admissions priorities, careful consideration of heterogeneous value-added can match students to schools that are better fits for them, and improve academic achievement.

3.7. Regression Results

Table 3.5: Impact of Attending Top Quality School Reduced Form Regressions

	Math PSAT	Math PSAT	English PSAT	English PSAT
Accept	-0.076***	-0.091***	0.019	0.006
	(0.019)	(0.024)	(0.019)	(0.023)
Hispanic x Accept	-0.095***	-0.057***	-0.072***	-0.042**
	(0.016)	(0.020)	(0.016)	(0.019)
Black x Accept	-0.072***	-0.015	-0.053***	0.012
	(0.020)	(0.026)	(0.019)	(0.025)
English Learner x Accept	-0.050	0.008	-0.211***	-0.192***
	(0.033)	(0.042)	(0.032)	(0.040)
Disability x Accept	-0.037	-0.063**	-0.070***	-0.105***
	(0.022)	(0.027)	(0.021)	(0.026)
Poverty x Accept	-0.041***	-0.055***	-0.078***	-0.085***
	(0.013)	(0.017)	(0.013)	(0.016)
Male x Accept	0.135***	0.077***	0.046***	0.014
	(0.013)	(0.016)	(0.012)	(0.015)
Middle School Tests x Accept	0.131***	0.062***	0.068***	0.015
	(0.010)	(0.012)	(0.009)	(0.011)
Hispanic	-0.317***	-0.368***	-0.188***	-0.225***
	(0.006)	(0.011)	(0.006)	(0.010)
Black	-0.420***	-0.462***	-0.293***	-0.332***
	(0.008)	(0.014)	(0.007)	(0.013)
English Learner	0.018**	0.008	-0.178***	-0.203***
	(0.009)	(0.016)	(0.008)	(0.015)
Disability	-0.068***	-0.063***	-0.012	0.011
	(0.007)	(0.013)	(0.007)	(0.013)
Poverty	-0.067***	-0.076***	-0.137***	-0.155***
	(0.006)	(0.010)	(0.006)	(0.010)
Male	0.138***	0.195***	0.023***	0.048***
	(0.005)	(0.009)	(0.005)	(0.008)
Middle School Tests	0.520***	0.571***	0.591***	0.633***
	(0.003)	(0.006)	(0.003)	(0.006)
Sample Size	79803.0	36077.0	79803.0	36077.0

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$. Columns 1 and 3 use the entire sample, and columns 2 and 4 subset to schools where at least 80% of Students took the PSAT.

Table 3.6: Impact of Attending Top Quality School IV Regressions

	Math PSAT	Math PSAT	English PSAT	English PSAT
Enroll	-0.031	-0.037	0.097***	0.107***
	(0.025)	(0.035)	(0.023)	(0.032)
Hispanic x Enroll	-0.115***	-0.089***	-0.103***	-0.082***
	(0.018)	(0.022)	(0.017)	(0.022)
Black x Enroll	-0.090***	-0.057**	-0.090***	-0.046
	(0.022)	(0.028)	(0.022)	(0.028)
English Learner x Enroll	-0.044	-0.011	-0.222***	-0.222***
	(0.037)	(0.045)	(0.035)	(0.044)
Disability x Enroll	-0.035	-0.077**	-0.078***	-0.134***
	(0.025)	(0.030)	(0.024)	(0.029)
Poverty x Enroll	-0.055***	-0.123***	-0.124***	-0.187***
	(0.015)	(0.020)	(0.014)	(0.019)
Male x Enroll	0.149***	0.056***	0.038***	-0.032
	(0.015)	(0.020)	(0.014)	(0.018)
Middle School Tests x Enroll	0.152***	0.047***	0.073***	-0.015
	(0.013)	(0.017)	(0.012)	(0.016)
Hispanic	-0.302***	-0.337***	-0.159***	-0.184***
	(0.006)	(0.012)	(0.006)	(0.011)
Black	-0.404***	-0.416***	-0.259***	-0.271***
	(0.007)	(0.014)	(0.007)	(0.014)
English Learner	0.015	0.029	-0.171***	-0.174***
	(0.008)	(0.016)	(0.008)	(0.015)
Disability	-0.072***	-0.049***	-0.008	0.039***
	(0.007)	(0.013)	(0.007)	(0.013)
Poverty	-0.057***	-0.010	-0.096***	-0.056***
	(0.006)	(0.011)	(0.005)	(0.010)
Male	0.127***	0.218***	0.032***	0.094***
	(0.005)	(0.011)	(0.005)	(0.010)
Middle School Tests	0.506***	0.588***	0.589***	0.662***
	(0.004)	(0.009)	(0.004)	(0.009)
Sample Size	78643	35102	78643	35102

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$. Columns 1 and 3 use the entire sample, and columns 2 and 4 subset to schools where at least 80% of Students took the PSAT.

Table 3.7: Impact of Attending Top Quality Screened School Reduced Form Regressions

	Math PSAT	Math PSAT	English PSAT	English PSAT
Accept	-0.072***	-0.072***	-0.003	-0.017
	(0.020)	(0.025)	(0.019)	(0.024)
Hispanic x Accept	-0.107***	-0.078***	-0.064***	-0.032
	(0.017)	(0.021)	(0.017)	(0.020)
Black x Accept	-0.077***	-0.026	-0.038	0.032
	(0.021)	(0.027)	(0.020)	(0.026)
English Learner x Accept	-0.051	0.003	-0.200***	-0.175***
	(0.034)	(0.044)	(0.033)	(0.042)
Disability x Accept	-0.030	-0.054	-0.050**	-0.074***
	(0.023)	(0.029)	(0.022)	(0.027)
Poverty x Accept	-0.055***	-0.083***	-0.065***	-0.073***
	(0.014)	(0.017)	(0.013)	(0.016)
Male x Accept	0.130***	0.066***	0.043***	0.008
	(0.013)	(0.016)	(0.013)	(0.015)
Middle School Tests x Accept	0.139***	0.069***	0.080***	0.028**
	(0.010)	(0.013)	(0.010)	(0.012)
Hispanic	-0.316***	-0.361***	-0.192***	-0.230***
	(0.006)	(0.010)	(0.006)	(0.010)
Black	-0.420***	-0.458***	-0.299***	-0.340***
	(0.007)	(0.013)	(0.007)	(0.013)
English Learner	0.018**	0.005	-0.180***	-0.210***
	(0.009)	(0.015)	(0.008)	(0.015)
Disability	-0.068***	-0.067***	-0.012	0.006
	(0.007)	(0.013)	(0.007)	(0.012)
Poverty	-0.070***	-0.075***	-0.147***	-0.170***
	(0.006)	(0.010)	(0.006)	(0.009)
Male	0.138***	0.195***	0.021***	0.045***
	(0.005)	(0.009)	(0.005)	(0.008)
Middle School Tests	0.520***	0.567***	0.594***	0.634***
	(0.003)	(0.006)	(0.003)	(0.005)
Sample Size	79803.0	36077.0	79803.0	36077.0

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$. Columns 1 and 3 use the entire sample, and columns 2 and 4 subset to schools where at least 80% of Students took the PSAT.

Table 3.8: Impact of Attending Top Quality Screened School IV Regressions

	Math PSAT	Math PSAT	English PSAT	English PSAT
Enroll	0.017	0.126***	0.106***	0.193***
	(0.024)	(0.029)	(0.022)	(0.027)
Hispanic x Enroll	-0.028	-0.053**	-0.021	-0.032
	(0.018)	(0.021)	(0.018)	(0.021)
Black x Enroll	0.006	-0.004	-0.004	0.019
	(0.022)	(0.027)	(0.023)	(0.028)
English Learner x Enroll	-0.033	0.021	-0.204***	-0.189***
	(0.037)	(0.045)	(0.034)	(0.042)
Disability x Enroll	-0.004	-0.019	-0.038	-0.066**
	(0.027)	(0.031)	(0.025)	(0.030)
Poverty x Enroll	-0.016	-0.065***	-0.071***	-0.109***
	(0.016)	(0.019)	(0.016)	(0.019)
Male x Enroll	0.108***	0.062***	0.022	-0.015
	(0.015)	(0.018)	(0.015)	(0.018)
Middle School Tests x Enroll	0.076***	0.041***	0.009	-0.025
	(0.013)	(0.015)	(0.012)	(0.015)
Hispanic	-0.276***	-0.262***	-0.141***	-0.131***
	(0.005)	(0.009)	(0.005)	(0.009)
Black	-0.382***	-0.353***	-0.246***	-0.231***
	(0.006)	(0.012)	(0.007)	(0.012)
English Learner	0.004	-0.014	-0.178***	-0.203***
	(0.008)	(0.014)	(0.008)	(0.014)
Disability	-0.084***	-0.080***	-0.016**	0.017
	(0.007)	(0.012)	(0.007)	(0.012)
Poverty	-0.075***	-0.059***	-0.108***	-0.094***
	(0.005)	(0.008)	(0.005)	(0.008)
Male	0.097***	0.130***	0.012**	0.037***
	(0.005)	(0.008)	(0.005)	(0.008)
Middle School Tests	0.467***	0.477***	0.561***	0.584***
	(0.003)	(0.006)	(0.003)	(0.006)
Sample Size	71241	28265	71241	28265

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$. Columns 1 and 3 use the entire sample, and columns 2 and 4 subset to schools where at least 80% of Students took the PSAT.

Table 3.9: Impact of Attending Other Screened School Reduced Form Regressions

	Math PSAT	Math PSAT	English PSAT	English PSAT
Accept	-0.047***	0.080***	-0.083***	-0.025
	(0.015)	(0.024)	(0.015)	(0.023)
Hispanic x Accept	0.126***	0.057***	0.122***	0.091***
	(0.012)	(0.019)	(0.011)	(0.018)
Black x Accept	0.097***	0.015	0.138***	0.075***
	(0.015)	(0.025)	(0.014)	(0.024)
English Learner x Accept	-0.006	-0.048	-0.064***	-0.089***
	(0.017)	(0.031)	(0.017)	(0.029)
Disability x Accept	-0.023	-0.041	-0.035**	-0.023
	(0.014)	(0.025)	(0.014)	(0.024)
Poverty x Accept	0.034***	0.011	0.043***	0.034**
	(0.012)	(0.018)	(0.011)	(0.017)
Male x Accept	0.021**	-0.024	0.038***	0.011
	(0.010)	(0.016)	(0.009)	(0.015)
Middle School Tests x Accept	-0.048***	-0.056***	-0.053***	-0.043***
	(0.006)	(0.011)	(0.006)	(0.010)
Hispanic	-0.448***	-0.454***	-0.299***	-0.306***
	(0.008)	(0.011)	(0.007)	(0.010)
Black	-0.535***	-0.525***	-0.408***	-0.398***
	(0.009)	(0.014)	(0.009)	(0.013)
English Learner	0.031***	0.042**	-0.154***	-0.177***
	(0.011)	(0.019)	(0.011)	(0.018)
Disability	-0.044***	-0.046***	0.008	0.008
	(0.009)	(0.014)	(0.008)	(0.013)
Poverty	-0.121***	-0.124***	-0.202***	-0.223***
	(0.007)	(0.010)	(0.007)	(0.009)
Male	0.144***	0.221***	0.010	0.043***
	(0.006)	(0.009)	(0.006)	(0.009)
Middle School Tests	0.606***	0.658***	0.659***	0.690***
	(0.004)	(0.006)	(0.004)	(0.006)
Sample Size	79803.0	36077.0	79803.0	36077.0

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$. Columns 1 and 3 use the entire sample, and columns 2 and 4 subset to schools where at least 80% of Students took the PSAT.

Table 3.10: Impact of Attending Other Screened School IV Regressions

	Math PSAT	Math PSAT	English PSAT	English PSAT
Enroll	0.014	0.084***	0.030	0.043
	(0.016)	(0.026)	(0.016)	(0.025)
Hispanic x Enroll	0.058***	0.015	0.044***	0.035
	(0.013)	(0.020)	(0.013)	(0.019)
Black x Enroll	0.012	-0.025	0.045***	0.003
	(0.016)	(0.026)	(0.016)	(0.026)
English Learner x Enroll	-0.019	-0.091***	-0.101***	-0.158***
	(0.019)	(0.031)	(0.018)	(0.030)
Disability x Enroll	-0.043***	-0.103***	-0.064***	-0.103***
	(0.015)	(0.025)	(0.015)	(0.025)
Poverty x Enroll	-0.002	-0.006	-0.036***	-0.033
	(0.012)	(0.020)	(0.012)	(0.020)
Male x Enroll	0.011	-0.057***	0.005	-0.047***
	(0.011)	(0.017)	(0.011)	(0.017)
Middle School Tests x Enroll	-0.047***	-0.101***	-0.070***	-0.109***
	(0.007)	(0.012)	(0.007)	(0.012)
Hispanic	-0.312***	-0.292***	-0.171***	-0.155***
	(0.008)	(0.010)	(0.008)	(0.010)
Black	-0.387***	-0.347***	-0.265***	-0.223***
	(0.009)	(0.013)	(0.009)	(0.013)
English Learner	0.033***	0.056***	-0.127***	-0.129***
	(0.012)	(0.018)	(0.012)	(0.017)
Disability	-0.039***	-0.008	0.025***	0.070***
	(0.009)	(0.014)	(0.009)	(0.014)
Poverty	-0.052***	-0.048***	-0.091***	-0.091***
	(0.007)	(0.009)	(0.007)	(0.009)
Male	0.121***	0.189***	0.024***	0.071***
	(0.006)	(0.009)	(0.006)	(0.009)
Middle School Tests	0.537***	0.581***	0.622***	0.660***
	(0.004)	(0.006)	(0.004)	(0.006)
Sample Size	71420	28382	71420	28382

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$. Columns 1 and 3 use the entire sample, and columns 2 and 4 subset to schools where at least 80% of Students took the PSAT.

Table 3.11: Impact of Attending Top Quality School Reduced Form Regressions

	Math Rank 9	English Rank 9	Math Rank 10	English Rank 10
Accept	-0.074***	-0.080***	-0.056***	-0.066***
	(0.007)	(0.007)	(0.008)	(0.008)
Hispanic x Accept	-0.030***	-0.051***	-0.042***	-0.046***
	(0.006)	(0.006)	(0.006)	(0.006)
Black x Accept	-0.048***	-0.063***	-0.047***	-0.055***
	(0.007)	(0.007)	(0.008)	(0.008)
English Learner x Accept	-0.053***	-0.010	-0.038***	-0.022
	(0.012)	(0.012)	(0.012)	(0.013)
Disability x Accept	-0.042***	-0.033***	-0.040***	-0.042***
	(0.008)	(0.008)	(0.008)	(0.008)
Poverty x Accept	-0.012**	-0.018***	-0.017***	-0.014***
	(0.005)	(0.005)	(0.005)	(0.005)
Male x Accept	-0.038***	-0.009	-0.032***	0.000
	(0.005)	(0.005)	(0.005)	(0.005)
Middle School Tests x Accept	-0.004	-0.005	-0.002	-0.001
	(0.004)	(0.004)	(0.004)	(0.004)
Hispanic	-0.017***	-0.019***	-0.019***	-0.020***
	(0.002)	(0.002)	(0.002)	(0.002)
Black	-0.010***	-0.021***	-0.006**	-0.015***
	(0.003)	(0.003)	(0.003)	(0.003)
English Learner	0.049***	0.053***	0.040***	0.045***
	(0.003)	(0.003)	(0.003)	(0.003)
Disability	-0.020***	-0.021***	-0.006**	-0.009***
	(0.002)	(0.002)	(0.003)	(0.003)
Poverty	-0.025***	-0.011***	-0.023***	-0.013***
	(0.002)	(0.002)	(0.002)	(0.002)
Male	-0.069***	-0.024***	-0.064***	-0.026***
	(0.002)	(0.002)	(0.002)	(0.002)
Middle School Tests	0.102***	0.116***	0.088***	0.095***
	(0.001)	(0.001)	(0.001)	(0.001)
Sample Size	104471.0	104316.0	100976.0	100849.0

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$.

Table 3.12: Impact of Attending Top Quality School IV Regressions

	Math Rank 9	English Rank 9	Math Rank 10	English Rank 10
Enroll	0.311***	0.288***	0.334***	0.311***
	(0.009)	(0.009)	(0.009)	(0.009)
Hispanic x Enroll	-0.250***	-0.262***	-0.260***	-0.255***
	(0.007)	(0.007)	(0.007)	(0.007)
Black x Enroll	-0.301***	-0.306***	-0.298***	-0.297***
	(0.008)	(0.008)	(0.008)	(0.008)
English Learner x Enroll	-0.150***	-0.101***	-0.134***	-0.112***
	(0.011)	(0.012)	(0.013)	(0.013)
Disability x Enroll	-0.102***	-0.090***	-0.101***	-0.101***
	(0.008)	(0.008)	(0.008)	(0.008)
Poverty x Enroll	-0.335***	-0.328***	-0.337***	-0.322***
	(0.005)	(0.005)	(0.006)	(0.006)
Male x Enroll	-0.191***	-0.153***	-0.182***	-0.141***
	(0.005)	(0.005)	(0.005)	(0.005)
Middle School Tests x Enroll	-0.073***	-0.071***	-0.070***	-0.066***
	(0.004)	(0.004)	(0.004)	(0.004)
Hispanic	0.195***	0.184***	0.191***	0.182***
	(0.003)	(0.003)	(0.003)	(0.003)
Black	0.238***	0.217***	0.240***	0.222***
	(0.003)	(0.003)	(0.003)	(0.003)
English Learner	0.148***	0.147***	0.137***	0.137***
	(0.003)	(0.003)	(0.004)	(0.004)
Disability	0.038***	0.034***	0.052***	0.047***
	(0.003)	(0.003)	(0.003)	(0.003)
Poverty	0.296***	0.296***	0.295***	0.292***
	(0.003)	(0.003)	(0.003)	(0.003)
Male	0.081***	0.118***	0.082***	0.113***
	(0.002)	(0.002)	(0.002)	(0.002)
Middle School Tests	0.170***	0.180***	0.154***	0.158***
	(0.002)	(0.002)	(0.002)	(0.002)
Sample Size	103444	103300	98877	98765

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$.

Table 3.13: Impact of Attending Top Quality Screened School Reduced Form Regressions

	Math Rank 9	English Rank 9	Math Rank 10	English Rank 10
Accept	-0.069***	-0.079***	-0.050***	-0.064***
	(0.008)	(0.007)	(0.008)	(0.008)
Hispanic x Accept	-0.034***	-0.055***	-0.047***	-0.051***
	(0.006)	(0.006)	(0.007)	(0.007)
Black x Accept	-0.045***	-0.059***	-0.043***	-0.050***
	(0.008)	(0.007)	(0.008)	(0.008)
English Learner x Accept	-0.054***	-0.014	-0.043***	-0.023
	(0.012)	(0.012)	(0.013)	(0.013)
Disability x Accept	-0.044***	-0.033***	-0.044***	-0.044***
	(0.008)	(0.008)	(0.009)	(0.009)
Poverty x Accept	-0.015***	-0.020***	-0.021***	-0.015***
	(0.005)	(0.005)	(0.005)	(0.005)
Male x Accept	-0.042***	-0.009	-0.037***	-0.000
	(0.005)	(0.005)	(0.005)	(0.005)
Middle School Tests x Accept	-0.004	-0.003	-0.002	-0.001
	(0.004)	(0.004)	(0.004)	(0.004)
Hispanic	-0.016***	-0.019***	-0.018***	-0.020***
	(0.002)	(0.002)	(0.002)	(0.002)
Black	-0.010***	-0.021***	-0.005**	-0.015***
	(0.003)	(0.003)	(0.003)	(0.003)
English Learner	0.049***	0.053***	0.040***	0.045***
	(0.003)	(0.003)	(0.003)	(0.003)
Disability	-0.021***	-0.022***	-0.007***	-0.010***
	(0.002)	(0.002)	(0.002)	(0.002)
Poverty	-0.022***	-0.009***	-0.020***	-0.011***
	(0.002)	(0.002)	(0.002)	(0.002)
Male	-0.068***	-0.023***	-0.063***	-0.025***
	(0.002)	(0.002)	(0.002)	(0.002)
Middle School Tests	0.101***	0.114***	0.086***	0.093***
	(0.001)	(0.001)	(0.001)	(0.001)
Sample Size	104471.0	104316.0	100976.0	100849.0

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$.

Table 3.14: Impact of Attending Top Quality Screened School IV Regressions

	Math Rank 9	English Rank 9	Math Rank 10	English Rank 10
Enroll	0.285***	0.263***	0.313***	0.283***
	(0.009)	(0.009)	(0.009)	(0.009)
Hispanic x Enroll	-0.266***	-0.287***	-0.282***	-0.279***
	(0.007)	(0.007)	(0.007)	(0.007)
Black x Enroll	-0.308***	-0.318***	-0.306***	-0.305***
	(0.008)	(0.008)	(0.009)	(0.009)
English Learner x Enroll	-0.151***	-0.101***	-0.135***	-0.109***
	(0.012)	(0.012)	(0.013)	(0.013)
Disability x Enroll	-0.100***	-0.083***	-0.102***	-0.097***
	(0.009)	(0.009)	(0.009)	(0.009)
Poverty x Enroll	-0.323***	-0.315***	-0.328***	-0.309***
	(0.006)	(0.006)	(0.006)	(0.006)
Male x Enroll	-0.175***	-0.146***	-0.171***	-0.141***
	(0.006)	(0.006)	(0.006)	(0.006)
Middle School Tests x Enroll	-0.060***	-0.048***	-0.057***	-0.048***
	(0.004)	(0.004)	(0.004)	(0.004)
Hispanic	0.195***	0.184***	0.191***	0.182***
	(0.003)	(0.003)	(0.003)	(0.003)
Black	0.237***	0.216***	0.240***	0.223***
	(0.003)	(0.003)	(0.003)	(0.003)
English Learner	0.150***	0.150***	0.138***	0.139***
	(0.003)	(0.003)	(0.004)	(0.003)
Disability	0.040***	0.037***	0.054***	0.049***
	(0.003)	(0.003)	(0.003)	(0.003)
Poverty	0.293***	0.294***	0.292***	0.289***
	(0.002)	(0.002)	(0.003)	(0.003)
Male	0.084***	0.119***	0.084***	0.114***
	(0.002)	(0.002)	(0.002)	(0.002)
Middle School Tests	0.172***	0.184***	0.155***	0.161***
	(0.002)	(0.001)	(0.002)	(0.002)
Sample Size	95335	95206	90862	90743

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$.

Table 3.15: Impact of Attending Other Screened School Reduced Form Regressions

	Math Rank 9	English Rank 9	Math Rank 10	English Rank 10
Accept	0.001	-0.001	-0.002	0.000
	(0.005)	(0.005)	(0.006)	(0.006)
Hispanic x Accept	-0.018***	-0.016***	-0.015***	-0.017***
	(0.004)	(0.004)	(0.004)	(0.004)
Black x Accept	-0.023***	-0.022***	-0.019***	-0.020***
	(0.005)	(0.005)	(0.005)	(0.005)
English Learner x Accept	0.029***	0.024***	0.015**	0.014**
	(0.006)	(0.006)	(0.006)	(0.006)
Disability x Accept	0.035***	0.041***	0.034***	0.030***
	(0.005)	(0.005)	(0.005)	(0.005)
Poverty x Accept	-0.006	-0.010**	-0.006	-0.010**
	(0.004)	(0.004)	(0.004)	(0.004)
Male x Accept	-0.003	-0.005	-0.004	-0.006
	(0.003)	(0.003)	(0.004)	(0.004)
Middle School Tests x Accept	0.013***	0.016***	0.013***	0.005**
	(0.002)	(0.002)	(0.002)	(0.002)
Hispanic	-0.006**	-0.010***	-0.012***	-0.012***
	(0.003)	(0.003)	(0.003)	(0.003)
Black	-0.003	-0.016***	-0.002	-0.010***
	(0.003)	(0.003)	(0.003)	(0.003)
English Learner	0.033***	0.041***	0.031***	0.037***
	(0.004)	(0.004)	(0.004)	(0.004)
Disability	-0.045***	-0.049***	-0.030***	-0.032***
	(0.003)	(0.003)	(0.003)	(0.003)
Poverty	-0.018***	-0.003	-0.018***	-0.006**
	(0.003)	(0.003)	(0.003)	(0.003)
Male	-0.072***	-0.022***	-0.066***	-0.023***
	(0.002)	(0.002)	(0.002)	(0.002)
Middle School Tests	0.086***	0.096***	0.072***	0.081***
	(0.001)	(0.001)	(0.001)	(0.001)
Sample Size	104471.0	104316.0	100976.0	100849.0

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$.

Table 3.16: Impact of Attending Other Screened School IV Regressions

	Math Rank 9	English Rank 9	Math Rank 10	English Rank 10
Enroll	0.505***	0.477***	0.499***	0.477***
	(0.007)	(0.006)	(0.007)	(0.007)
Hispanic x Enroll	-0.259***	-0.243***	-0.250***	-0.242***
	(0.005)	(0.005)	(0.006)	(0.006)
Black x Enroll	-0.307***	-0.289***	-0.298***	-0.287***
	(0.006)	(0.006)	(0.007)	(0.007)
English Learner x Enroll	-0.071***	-0.074***	-0.089***	-0.087***
	(0.007)	(0.007)	(0.008)	(0.008)
Disability x Enroll	-0.042***	-0.031***	-0.042***	-0.045***
	(0.006)	(0.006)	(0.006)	(0.006)
Poverty x Enroll	-0.327***	-0.322***	-0.327***	-0.316***
	(0.005)	(0.005)	(0.005)	(0.005)
Male x Enroll	-0.153***	-0.147***	-0.154***	-0.148***
	(0.004)	(0.004)	(0.005)	(0.005)
Middle School Tests x Enroll	-0.080***	-0.075***	-0.081***	-0.089***
	(0.003)	(0.003)	(0.003)	(0.003)
Hispanic	0.212***	0.195***	0.204***	0.194***
	(0.004)	(0.004)	(0.004)	(0.004)
Black	0.255***	0.228***	0.255***	0.236***
	(0.004)	(0.004)	(0.004)	(0.004)
English Learner	0.135***	0.141***	0.135***	0.138***
	(0.005)	(0.005)	(0.005)	(0.005)
Disability	0.032***	0.025***	0.048***	0.045***
	(0.004)	(0.004)	(0.004)	(0.004)
Poverty	0.284***	0.289***	0.285***	0.283***
	(0.003)	(0.003)	(0.003)	(0.003)
Male	0.080***	0.118***	0.083***	0.114***
	(0.003)	(0.003)	(0.003)	(0.003)
Middle School Tests	0.188***	0.198***	0.173***	0.182***
	(0.002)	(0.002)	(0.002)	(0.002)
Sample Size	95515	95385	91041	90923

Note: Standard Errors are in parentheses. *** $p < .01$, ** $p < .05$, * $p < .01$.

APPENDIX A

APPENDIX A: SUPPLEMENTARY MATERIAL FOR WELFARE PROPERTIES OF THE TAIWAN MECHANISM

A.1. Proof of Theorem 1.1

Theorem 1.1. *An equilibrium exists.*

Proof. The proof proceeds in three steps. First, I define a fixed point in the choice density vector, c. Then, I show that c has a fixed point. Lastly, I show that if c has a fixed point, an equilibrium must exist.

Defining a Fixed Point:

Definition A.1. *The* ***best response set****,* $\Sigma(c)$*, is the set of strategies,* σ*, which best respond to choice density vector* c

Define a correspondence $c^* : \mathcal{C} \to \mathcal{C}$, such that $d \in c^*(c)$ if and only if d is the choice density vector induced by some $\sigma \in \Sigma(c)$. The choice density vector, c, is a fixed point if $c \in c^*(c)$.

Existence of Fixed Point:

The correspondence, c^*, maps the closed and bounded set $\mathcal{C}$ to itself. Lemma A.2, proven in the appendix, shows that c^* is closed-valued and convex-valued. Lemma A.3, proven in the appendix, shows that c^* is an upper hemicontinuous correspondence. Therefore, c^* has a fixed point by the Kakutani fixed-point theorem.

Existence of Fixed Point Implies the existence of an equilibrium

I have shown there exists c such that $c \in c^*(c)$. It follows that there exists $\sigma \in \Sigma(c)$, such that $c(\sigma) = c$. The strategy, σ, is a best response to c by the definition of $\Sigma(c)$. The strategy, σ, is, therefore, an equilibrium. □

A.2. Proof of Theorem 1.3

Theorem 1.3. *Any value of $\lambda > 0$ weakly ex ante Pareto dominates $\lambda = 0$ for sophisticated students.*

Proof. When $\lambda = 0$, the Taiwan mechanism is equivalent to deferred acceptance, so true preference revelation is dominant strategy incentive compatible. Lemma A.5 shows that students with $v_1 > v_2 > \ldots > v_N$ submit the rank order list $r = (1, 2, \ldots, N)$ in any equilibrium of deferred acceptance. Students with $v_s = v_{s+1}$ for some school s may have equal utility from $r = (1, 2, \ldots, N)$ and some alternatives, but there are measure 0 such students, so their choices do not affect the equilibrium outcome. Ex ante, students who submit the same rank order list are treated equally at each school, so all of the measure 1 students who submit $r = (1, 2, \ldots, N)$ have the same vector of probabilities of admission. The unique equilibrium vector of probabilities of admission when $\lambda = 0$, conditional on submitting $r = (1, 2, \ldots, N)$ is, therefore, the schools' capacity: $p_s^{(1,2,\ldots,N)}(\lambda = 0) = (q_1, \ldots, q_{N-1}, 1 - \sum_1^{N-1} q_s)$. Furthermore, every student's unique equilibrium expected utility is simply their utility vector multiplied by the vector of the schools' capacity: $v \cdot (q_1, \ldots, q_{N-1}, 1 - \sum_1^{N-1} q_s)$.

Suppose $\lambda \neq 0$. Fix an equilibrium strategy, σ, and an associated choice density vector, c. Recall that $P^r(c, \lambda)$ denotes the vector of probabilities of admission for students who submit r induced by c and λ and $P(c, \lambda)$ is the matrix of probabilities of admission where each column is $P^r(c, \lambda)$ for some r:

$$P(c,\lambda) = \begin{pmatrix} p_1^{r_1}(c,\lambda) & \cdots & \cdots & \cdots & p_1^{r_{|\mathcal{R}|}}(c,\lambda) \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ p_N^{r_1}(c,\lambda) & \cdots & \cdots & \cdots & p_N^{r_{|\mathcal{R}|}}(c,\lambda) \end{pmatrix}.$$

Lemma A.6 shows that in equilibrium all schools other than the outside option are at full capacity, which implies that $P(c, \lambda) \cdot c = (q_1, \ldots, q_{N-1}, 1 - \sum_1^{N-1} q_s)$. Sophisticated students only submit

rank order list r^* if

$$v \cdot P^{r^*}(c, \lambda) \geq v \cdot P^{r}(c, \lambda) \ \ \forall r \in \mathcal{R}.$$

Multiplying each side of each inequality by c^r and summing all inequalities yields

$$v \cdot P^{r^*}(c, \lambda) \geq v \cdot (P(c, \lambda) \cdot c) = v \cdot (q_1, \ldots, q_{N-1}, 1 - \sum_{1}^{N-1} q_s).$$

The right-hand side is the equilibrium expected utility of a student with utility vector v when $\lambda = 0$, so the inequality demonstrates that the student's utility is weakly greater when $\lambda \neq 0$. □

A.3. No General Pareto Ranking

This section shows that any 3 school case can serve as a counter-example to demonstrate that higher deduction rules do not necessarily weakly Pareto dominate lower deduction rules. Theorem A.1 demonstrates that in the 3 school case, it is impossible to Pareto rank any deduction rules that are not equivalent to deferred acceptance.

Theorem A.1. *If there are three schools, $\lambda = (0, \lambda_2), \lambda_2 \in (0, \lambda_2^{Boston})$, and $c^{(2)}(\lambda) \neq 0$, then there exists a v_2^*, such that the expected utility of a student with $v_2 = v_2^*$ is higher at $\lambda_2 - \epsilon$ than at λ_2 for some $\epsilon > 0$.*

Proof. I will show that students who are indifferent between ranking school 1 first and school 2 first at λ_2 have strictly decreasing marginal utility on λ_2 at λ_2. The implication is that these students have higher utility at a $\lambda_2' < \lambda_2$. If there are no students who are indifferent between ranking School 1 first and School 2 first at λ the proof of the theorem is trivial. A marginal increase in λ_2 will not induce any students to rank School 2 first, so all students who rank School 1 first are harmed because their probability of admission at School 2 decreases, and their probability of admission at School 1 remains the same. The reason their probability of admission at School 2 decreases is that some students rank School 2 first ($c^{(2)}(\lambda_2) \neq 0$) and when λ_2 increases, students who rank School 1 first are put at a larger disadvantage for admissions at School 2 relative to students who rank School 2 first. If $c^{(2)}(\lambda_2) = 0$ then all students submit the same ranking and the deduction rule has

no effect. In this case, raising λ_2 can be a Pareto improvement for the same reason that raising λ_2 from $\lambda_2 = 0$ is a Pareto improvement.

First, I identify students who are indifferent between ranking School 1 first and School 2 first. Equation (1.6) shows that a student is indifferent between ranking School 1 first and School 2 first if

$$(1 - \bar{L}_1)v_1 + (\bar{L}_1 - \bar{L}_2 - \lambda_2)v_2 = (1 - \bar{L}_2)v_2.$$

Some algebra, which uses Equation (1.5) to plug in for $\bar{L}$ yields:

$$v_2 = \frac{q_1 v_1}{q_1 + \lambda_2 c^{(1,2)}}.$$

I plug in for $\bar{L}$ in Equation (1.6) to solve for the student's utility from ranking School 1 first at λ_2:

$$\frac{q_1}{c^{(1,2)}}(v_1 - v_2) + \frac{\lambda_2 c^{(1,2)}}{c^{(1,2)} + c^{(2)}}v_2 - \lambda_2 v_2 + \frac{q_1 + q_2}{c^{(1,2)} + c^{(2)}}v_2.$$

Because there are only three schools, $c^{(1,2)} + c^{(2)} = 1$:

$$\frac{q_1}{c^{(1,2)}}(v_1 - v_2) - \lambda_2 c^{(2)} v_2 + (q_1 + q_2)v_2.$$

I take the derivative with respect to λ_2. Lemma A.9 shows that the equation is differentiable.

$$-\frac{q_1}{{c^{(1,2)}}^2}\frac{\partial c^{(1,2)}}{\partial \lambda_2}(v_1 - v_2) - v_2(c^{(2)} + \lambda_2 \frac{\partial c^{(2)}}{\partial \lambda_2}).$$

I divide by v_2:

$$-\frac{q_1}{{c^{(1,2)}}^2}\frac{\partial c^{(1,2)}}{\partial \lambda_2}(\frac{v_1}{v_2} - 1) - (c^{(2)} + \lambda_2 \frac{\partial c^{(2)}}{\partial \lambda_2}).$$

Note that $\frac{v_1}{v_2} - 1 = \frac{v_1(q_1 + \lambda_2 c^{(1,2)})}{q_1 v_1} - 1 = \frac{q_1 + \lambda_2 c^{(1,2)}}{q_1} - 1 = \frac{\lambda_2 c^{(1,2)}}{q_1}$. Substituting yields:

$$-\frac{\lambda_2}{c^{(1,2)}}\frac{\partial c^{(1,2)}}{\partial \lambda_2} - c^{(2)} - \lambda_2 \frac{\partial c^{(2)}}{\partial \lambda_2}.$$

Because there are only 3 schools, $-\frac{\partial c^{(1,2)}}{\partial \lambda_2} = \frac{\partial c^{(2)}}{\partial \lambda_2}$. I plug in for $-\frac{\partial c^{(1,2)}}{\partial \lambda_2}$:

$$\frac{\partial c^{(2)}}{\partial \lambda_2}(\frac{\lambda_2}{c^{(1,2)}} - \lambda_2) - c^{(2)}$$

Multiplying by $c^{(1,2)}$ yields:

$$\frac{\partial c^{(2)}}{\partial \lambda_2}\lambda_2(1 - c^{(1,2)}) - c^{(2)}c^{(1,2)}$$

Dividing by $c^{(2)}$ and using the fact that $1 - c^{(1,2)} = c^{(2)}$ yields:

$$\frac{\partial c^{(2)}}{\partial \lambda_2}\lambda_2 - c^{(1,2)}.$$

This term is strictly less than zero. The reason is that sophisticated students choose to apply to School 2 if $v_2 > \frac{q_1 v_1}{q_1 + \lambda_2 c^{(1,2)}}$. The derivative of the denominator with respect to λ_2 is $-\lambda_2 \frac{\partial c^{(2)}}{\partial \lambda_2} + c^{(1,2)}$. If $-\lambda_2 \frac{\partial c^{(2)}}{\partial \lambda_2} + c^{(1,2)} < 0$, $\frac{\partial c^{(2)}}{\partial \lambda_2} < 0$, a clear contradiction. It follows that $-\lambda_2 \frac{\partial c^{(2)}}{\partial \lambda_2} + c^{(1,2)} > 0$. Multiplying by negative 1 yields the required result. □

A.4. Proof of Theorem 1.4

This section continues the proof of Theorem 1.4 by taking the derivative of Equation (1.8) with respect to λ at $\lambda = 0$. Lemma A.7 shows that $\frac{\partial c}{\partial \lambda}$ exists at the equilibrium c when $\lambda = 0$, and the differentiability of Equation (1.8) with respect to λ follows immediately. For convenience, I reproduce and slightly reorganize Equation (1.8) here:

$$\sum_{s=1}^{N} \left[\overbrace{\bar{L}_{s-1}}^{\text{Term 1}} - \overbrace{\frac{\sum_{r:s\in r} c^r \bar{L}_t}{\sum_{r:s\in r} c^r}}^{\text{Term 2}} + \overbrace{\frac{q_s}{\sum_{r:s\in r} c^r}}^{\text{Term 3}} - \overbrace{\frac{\sum_{r:s\in r} c^r(\lambda_{r(t)} - \lambda_{r(s)})}{\sum_{r:s\in r} c^r}}^{\text{Term 4}} - \overbrace{(\lambda_s - \lambda_{s-1})}^{\text{Term 5}} \right]^+ E[v_s]$$

I separate the derivative with respect to λ into mechanical and behavioral effects. Recall that the mechanical effect holds c constant and allows λ and $\bar{L}$ to vary, whereas the behavioral effect

holds λ and $\bar{L}$ constant and allows c to vary.

$$\text{Mechanical Effect:} \sum_{s=1}^{N} \Big[\overbrace{\partial \bar{L}_{s-1}}^{\text{Term 1: mechanical}} - \overbrace{\partial \bar{L}_{s-1}}^{\text{Term 2: mechanical}} + \overbrace{\partial(\lambda_s - \lambda_{s-1})}^{\text{Term 4: mechanical}} - \overbrace{\partial(\lambda_s - \lambda_{s-1})}^{\text{Term 5: mechanical}} \Big] E[v_s]$$

$$\text{Behavioral Effect:} \sum_{s=1}^{N} \left[\overbrace{-\sum_{r:s\in r} \partial c^r (\bar{L}_t - \bar{L}_{s-1})}^{\text{Term 2: behavioral}} + \overbrace{-q_s \sum_{r:s\in r} \partial c^r}^{\text{Term 3: behavioral}} + \overbrace{0}^{\text{Term 4: behavioral}} \right] E[v_s]$$

The derivatives of terms 2 and 4 are not trivial, so I pause to explain here.

$$\text{Term 2: } -\partial \frac{\sum_{r:s\in r} c^r \bar{L}_t}{\sum_{r:s\in r} c^r} = \overbrace{-\partial \bar{L}_{s-1}}^{\text{Term 2: mechanical}} \overbrace{-\sum_{r:s\in r} \partial c^r (\bar{L}_t - \bar{L}_{s-1})}^{\text{Term 2: behavioral}}$$

The choice density vector, c and the cutoff lottery numbers, $\bar{L}$, both vary with respect to λ, so taking the derivative requires the product rule. The mechanical effect takes the derivative of $\bar{L}$, holding c^r constant. Measure 1 students reveal their preferences truthfully at $\lambda = 0$ (Lemma A.5), so $c^{(1,2,\ldots,N)} = 1$ and any other c is 0. The mechanical effect is, therefore, the derivative with respect to λ of $-\frac{1\cdot \bar{L}_{s-1}}{1} = -\partial \bar{L}_{s-1}$. The behavioral effect differentiates with respect to c^r, holding $\bar{L}_t$ constant. I use the quotient rule:

$$-\partial \frac{\sum_{r:s\in r} c^r \bar{L}_t}{\sum_{r:s\in r} c^r} = -\frac{\sum_{r:s\in r} \partial c^r \bar{L}_t \sum_{r:s\in r} c^r - \sum_{r:s\in r} \partial c^r \sum_{r:s\in r} c^r \bar{L}_t}{(\sum_{r:s\in r} c^r)^2} = -\sum_{r:s\in r} \partial c^r (\bar{L}_t - \bar{L}_{s-1}).$$

The last equality follows from the fact that $c^{(1,2,\ldots,N)} = 1$, and $c^r = 0$ for any other r.

Next, I explain how to compute the derivative of term 4.

$$\text{Term 4: } \partial \frac{\sum_{r:s\in r} c^r (\lambda_{r(t)} - \lambda_{r(s)})}{\sum_{r:s\in r} c^r} = \partial \overbrace{(\lambda_s - \lambda_{s-1})}^{\text{Term 4: mechanical}} + \overbrace{0}^{\text{Term 4: behavioral}}$$

The mechanical effect holds c^r constant. As before, all students submit $r = (1, 2, \ldots, N)$. The mechanical effect is, therefore, the derivative with respect to λ of $-1 \cdot (\lambda_{s-1} - \lambda_s) = \partial(\lambda_s - \lambda_{s-1})$.

The behavioral effect holds λ constant at 0: $\frac{\sum_{r:s\in r} c^r * 0}{\sum_{r:s\in r} c^r} = 0$.

The above equations demonstrate that the mechanical effect is 0 and I explained the intuition for this result in the main text of the paper. All that remains is for me to show that the behavioral effect is positive. The change in the mass of students who include s in their rank order list is the negative of the change in the mass of students who exclude s from their rank order list, so $\sum_{r:s\in r} \partial c^r = -\sum_{r:s\notin r} \partial c^r$. I rewrite the behavioral effect, using that substitution for the term $-q_s \sum_{r:s\in r} \partial c^r$.

$$\text{Behavioral Effect:} \sum_{s=1}^{N} \left[-\sum_{r:s\in r} \partial c^r (\bar{L}_t - \bar{L}_{s-1}) + q_s \sum_{r:s\notin r} \partial c^r \right] E[v_s]$$

Recall that at $\lambda = 0$ all students rank the schools in order, so $\bar{L}_t = 1 - \sum_{s=1}^{t} q_t$. It follows that $(\bar{L}_t - \bar{L}_{s-1}) = \sum_{i=t+1}^{s-1} q_i$. I substitute in now:

$$\text{Behavioral Effect:} \sum_{s=1}^{N} \left[-\sum_{r:s\in r} \partial c^r \sum_{i=t+1}^{s-1} q_i + q_s \sum_{r:s\notin r} \partial c^r \right] E[v_s]$$

Note that $\sum_{s=1}^{N} -\sum_{r:s\in r} (\partial c^r \sum_{i=t+1}^{s-1} q_i) E[v_s] = \sum_{s=1}^{N} -\sum_{r:s\notin r} (\partial c^r q_s) E[v_k]$ where k is the school with the lowest index such that $k \in r$ and $k > s$. Substituting in yields:

$$\text{Behavioral Effect:} \sum_{s=1}^{N} (q_s \sum_{r:s\notin r} \partial c^r (E[v_s] - E[v_k])).$$

The expression is strictly positive because $s < k \rightarrow E[v_s] > E[v_k]$, and Lemma A.8 shows that if f has full support, then $\sum_{r:s\notin r} \partial c^r > 0$ for some s. Lastly, the upper hemicontinuity of c^*, (Lemma A.3) implies that if the derivative of sincere student's utility is positive at $\lambda = 0$, then sincere student's utility is strictly greater at any equilibrium of the Taiwan mechanism for λ sufficiently small. □

The intuition is that students who omit School s in their rank order list exchange q_s probability of admission to s for an additional q_s probability of admission to k. Sincere students receive the

additional q_s probability of admission to School s multiplied by the measure of students who omit s from their ranking in exchange for the equivalent probability of admission at some $k > s$. They are strictly better off because they prefer School s to School k.

A.5. Example with Multiple Equilibria

There are 4 schools each with capacity $q_i = .25$. There are 3 groups of students and the following table summarizes their preferences.

Table A.1: Distribution of Students

Group	A	B	C
Mass	.25	.25	.5
Preferences			
s_1	1	1	1
s_2	0	0.4	0.5
s_3	0	0	0.5
s_4	0	0	0

Let λ be sufficiently large such that the Taiwan mechanism is equivalent to the Boston mechanism. I demonstrate the existence of two equilibria with distinct probabilities of admission to each school. The below tables describe the proposed equilibria.

Table A.2: Equilibrium 1 Rank Order Lists and Probabilities of Admission

Eq. 1	A	B	C
1st Choice	1	2	1
2nd Choice	3	3	3
3rd Choice	4	4	4
Prob Admit	s_1	s_2	s_3
1st Choice	$\frac{1}{3}$	1	1
2nd Choice	0	0	$\frac{1}{2}$
3rd Choice	0	0	0

Probabilities are conditional on rejection from earlier choices.

Table A.3: Equilibrium 2 Rank Order Lists and Probabilities of Admission

Eq. 2	A	B	$\frac{1}{2}C$	$\frac{1}{2}C$
1st Choice	1	1	2	3
2nd Choice	4	4	4	4
Prob	s_1	s_2	s_3	
1st Choice	$\frac{1}{2}$	1	1	
2nd Choice	0	0	0	
Prob admit	0	0	0	

Probabilities are conditional on rejection from earlier choices.

First, I verify equilibrium 1. Students in group A are obviously playing the optimal strategy. Students in group B are guaranteed admission to school 2, yielding .4 utils. Deviating to ranking School 1 first is not profitable. A student who chooses this deviation has $\frac{1}{3}$ probability of admission to School 1, and no chance of admission to School 2, so their utility is only $\frac{1}{3}$. Students in group B have utility $\frac{1}{3}(1 + .5) = .5$; deviating to ranking School 2 or School 3 first yields the same utility.

Next, I verify equilibrium 2. Students in group A are obviously playing the optimal strategy. Students in group B get $\frac{1}{2}$ utils, so they cannot possibly prefer to rank School 2 first. Students in group C get $\frac{1}{2}$ utils regardless of whether they rank School 2 or School 3 first. If instead a student chooses to rank School 1 first, they get the same $\frac{1}{2}$ utils.

A.6. Lemmas

Lemma A.1. *Let $\Sigma(c)$ denote the set of strategies that best respond to choice density vector c. For any c, there exists $\sigma \in \Sigma(c)$ such that $\sigma(v)$ is a monotonically increasing rank order list for all v.*

Proof. Let $r \in \sigma(v)$, and suppose r is not monotonically increasing, so $r = (\ldots, t, s, \ldots), t > s$. Consider the last t, s pair in r such that this is true. Suppose that $\bar{L}_t < \bar{L}_s$. The student must be rejected at t before applying to s, so the student's probability of admission at s is 0. Omitting s from the student's rank order list weakly increases the probability of admission at schools ranked after s because λ is weakly increasing, so omitting s provides weakly greater utility.

Alternatively, suppose $\bar{L}_t > \bar{L}_s$ (even though this is impossible in equilibrium). If the probability of admission at t is 0 then omitting t is trivially rationalizable. If the probability of admission at t is positive, Equation 1.6 shows that omitting t transfers the probability of admission from t to s and weakly increases the probability of admission to s' or a school ranked ahead of s' for any s' ranked after s in r. Because t and s are the last non monotonic pair, schools ranked ahead of s' are weakly preferred to s'. Omitting t, therefore, weakly increases the student's utility.

The rank order list r is finite, so this procedure eventually yields a monotonically increasing rank order list which generates weakly greater utility than r.

□

Lemma A.2. *c^* is closed-valued and convex-valued.*

Proof. I prove the lemma by characterizing $c^*(c)$ for an arbitrary c. I will show that:

$$d \in c^*(c) \iff d^{[r]} = (1 - p_{soph})\mathbb{I}(r = (1, \ldots, N) \in [r]) + p_{soph} \int_{V^{[r]}} f(v),\ dv,$$
$$\text{and } d^{(1,\ldots,N)} \geq 1 - p_{soph}.$$

where the equivalence class $[r]$ is the set $\{r' \in \mathcal{R} : P^{r'}(c) = P^r(c)\}$, and $V^{[r]}$ is the set of v for which it is rational to choose a rank order list from the equivalence class $[r]$:

$$v \in V^{[r]} \iff v \cdot (P^r(c) - P^{r'}(c)) \geq 0 \ \ \forall r' \in \mathcal{R}.$$

The set of v which are in $V^{[r]}$ and $V^{[r']}$ for $[r'] \neq [r]$ is measure 0. To see this, observe that in order to be in $V^{[r]}$ and $V^{[r']}$, v must be the solution to the following homogeneous system of equations: $v \cdot (P^r - P^{r'}) = 0,\ P^r \neq P^{r'}$, and the set of solutions to a homogeneous system of equations is measure 0.

Any σ which best responds to $P(c)$, therefore, must map a mass of students equal to $\int_{V^{[r]}} f(v)$ to each equivalence class. I denote this mass, $c^{[r]} = \sum_{r' \in [r]} c^{r'}$. Recall that σ is only responsible for

the decision making of the sophisticated fraction of students, p_{soph}, and that the remaining $1-p_{soph}$ students always submit $r = (1, \ldots, N)$. The if portion of the characterization follows immediately:

$$\text{If } d \in c^*(c) \text{ then } d^{[r]} = (1 - p_{soph})\mathbb{I}(r = (1, \ldots, N) \in [r]) + p_{soph} \int_{V^{[r]}} f(v),\ dv,$$
$$\text{and } d^{(1,\ldots,N)} \geq 1 - p_{soph}.$$

The continuity of $\int_{V^{[r]}} f(v)$ ensures that that this mass can divided arbitrarily between the c^r that comprise $c^{[r]}$ which demonstrates the only if portion of the characterization. The characterization shows that $c^*(c)$ is finitely many linear equations and inequalities and, therefore, closed and convex.

□

Lemma A.3. *$c^*(c)$ is an upper hemicontinuous correspondence.*

Proof. The proof proceeds in 2 steps. First I show that the matrix of admissions probabilities, P, is a continuous function of the choice density vector, c. I then show that c^* is an upper hemicontinuous correspondence of P.

Step 1: P is a continuous function of c.

I prove that $P(c)$ is continuous by induction. The induction hypothesis is that if p_t^r is a continuous functions of c for all $t < s$ and all r, then p_s^r is a continuous function of c for all r. First, I demonstrate the base case:

$$p_1^r = \begin{cases} min(\frac{q_1}{\sum_{r:1\in r} c^r}, 1) & \text{if } r(1) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

The probability of admission rank order list r generates at school 1, p_1^r, is clearly a continuous function of c for all r.

Next, I demonstrate the inductive step by examining how $\bar{L}_s$ varies as a function of c. Recall

Equation (1.3):

$$T_s - q_s = \sum_r w_s^r U[-\lambda_{r(s)}, 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}](\bar{L}_s).$$

I plug in for T_s and w_s^r and rearrange:

$$0 = \sum_{r:s\in r} c^r(1 - \sum_{t<s} p_t^r)U[-\lambda_{r(s)}, 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}](\bar{L}_s) - \sum_{r:s\in r} c^r(1 - \sum_{t<s} p_t^r) - q_s.$$

I can write the above equation as a $G(c, \bar{L}_s) = 0$, where G is a continuous function of c because $\sum_{t<s} p_t^r$ is a continuous function of c by the induction hypothesis. It follows that for any $\delta > 0$, there exists $a > 0$ such that if $|c' - c| < a \rightarrow |G(c', \bar{L}_s)| < \delta$.

Suppose $\bar{L}_s(c)$ is in the interior of the distribution of adjusted lottery numbers, $[-\lambda_{r(s)}, 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}]$ for some r. Then the derivative of G with respect to $\bar{L}_s$ is bounded below by $w_s^r \frac{1}{1-\sum_{t<s} p_t^r}$, which is the PDF of adjusted lottery numbers induced by r, weighted by the number of students who submit r. It follows that $|\bar{L}_s(c') - \bar{L}_s(c)| < \frac{\delta}{w_s^r \frac{1}{1-\sum_{t<s} p_t^r}}$ as long as δ is sufficiently small such that $\bar{L}_s(c')$ remains in the interior of the distribution of adjusted lottery numbers induced by r. Clearly, $\bar{L}_s(c)$ is a continuous function of c when $\bar{L}_s(c)$ is in the interior of the distribution of adjusted lottery numbers for some r.

Alternatively, suppose $\bar{L}_s(c)$ is not in the interior of the distribution of adjusted lottery numbers, $[-\lambda_{r(s)}, 1 - \sum_{t<s} p_t^r - \lambda_{r(s)}]$, for any r. This implies that every rank order list either yields no chance of admission to s, or guarantees admission to s conditional on rejection from earlier ranked schools. Because $\bar{L}_s(c)$ is the minimum lottery number needed for admission, if it is not in the interior of any distribution, it must be the left bound of one of the distributions that guarantee admission to s. This implies that $\bar{L}_s(c) = -\lambda_{r(s)}$ for some r.

Now consider a marginal change from c to c'. The marginal change must either cause $\bar{L}_s$ to increase or decrease. First suppose it causes $\bar{L}_s$ to increase. Then the derivative of G with respect to $\bar{L}_s$ is bounded below by $w_s^r \frac{1}{1-\sum_{t<s} p_t^r}$ and the above argument shows that $\bar{L}_s$ varies continuously. Suppose instead that the marginal change from c to c' causes $\bar{L}_s$ to decrease. The minimum adjusted

lottery number for admission to s immediately drops to $1 - \sum_{t<s} p_t^{r'} - \lambda_{r'(s)}$, where r' is the rank order list with the largest upper bound on the distribution of lottery numbers, among rank order lists with upper bounds less than $\bar{L}_s(c)$. After dropping to $1 - \sum_{t<s} p_t^{r'} - \lambda_{r'(s)}$, however, the above reasoning demonstrates that $\bar{L}_s$ again varies continuously. Equation (1.6) shows that in both cases $p_s^r(c)$ is a continuous function of c for all r.

Step 2: c^* is an upper hemicontinuous correspondence of $P(c)$.

Recall from the proof of *Lemma A.2* that:

$$v \in V^{[r]} \iff v \cdot (P^r(c) - P^{r'}(c)) \geq 0 \ \ \forall r' \in \mathcal{R}.$$

This inequality defines a hyperplane for each $P^{r'}(c)$, and $V^{[r]}$ is a convex polyhedron whose boundaries are given by this finite collection of hyperplanes. I examine how each hyperplane changes as $P^r(c) - P^{r'}(c)$ changes. Let $a = P^r(c) - P^{r'}(c)$, and consider a vector $a' = a_1 + \delta_1, \ldots, a_n + \delta_n$ where $|\delta_i| \leq \Delta$. Suppose v' is a solution to the inequality $v' \cdot a' \geq 0$. Then v' is also a solution to the inequality $v' \cdot a \geq -v' \cdot \delta$. I can bound the right hand side above by $N \cdot \Delta$ and below by $-N \cdot \Delta$ because $v_i' \leq 1$ and $|\delta_i| < \Delta$. Without loss of generality, I assume that $-v' \cdot \delta > 0$, because the proof is symmetric if $-v' \cdot \delta < 0$.

I have shown that any v' that solves the inequality $v \cdot a' \geq 0$ also solves the inequality $v \cdot a \geq N \cdot \Delta$, which also defines a hyperplane. The added volume from the δ change to $P^r(c) - P^{r'}(c)$ is bounded above by the volume in-between the two hyperplanes: $v \cdot a \geq N \cdot \Delta$ and $v \cdot a \geq 0$. If f is uniform, then Ball's theorem states that the maximal cross section of a hypercube is $\sqrt{2}$, and V is a subset of the hypercube. The volume between the two hyperplanes is, therefore, bounded above by $\sqrt{2}N\Delta$ when f is uniform. Of course, f need not be uniform. Let $\bar{f}$ represent an upper bound of $f(v)$. Then the total change to c^r from perturbing this hyperplane is bounded above by $\sqrt{2}N\Delta\bar{f}$.

I started with fewer than N! hyperplanes and I have shown that perturbing each hyperplane adds at most $\sqrt{2}N\Delta\bar{f}$ volume to $V^{[r]}$. Even if the area added by all of the hyperplanes is disjoint, the maximum change to the volume of $V^{[r]}$ from perturbing $P^{[r]} - P^{r'}$ for each $r' \notin [r]$ is $\sqrt{2}N!N\Delta\bar{f}$.

Any new P' in a Δ neighborhood of P can perform this series of perturbations once for each element in $|r|$. The upper bound on the change to $V^{[r]}$ at any P' within a Δ neighborhood of P is, therefore, $\sqrt{2}N^2\Delta\bar{f}|[r]|$.

I also choose Δ sufficiently small such that no equivalence class can gain elements under any P'. This restriction is necessary because if $[r]$ gains the element $[r']$, then V^r can be mapped to r' rather than r, which may cause $c^{|r|}$ to lose mass discontinuously. Observe that $c^{|r|}$ is unaffected when $|r|$ splinters into multiple equivalence classes because $V^{[r]}$ still must be mapped to elements of the original equivalence class, $[r]$.

The above reasoning, as well as the characterization of $c^*(c)$ derived in Lemma A.2, show that any $d \in c^*(P')$ satisfies the following inequality, where the equivalence class $|r|$, and set $V^{[r]}$ are defined by P rather than P':

$$|d^{[r]} - (1 - p_{soph})\mathbb{I}(r = (1, \ldots, N) \in [r]) + p_{soph} \int_{V^{[r]}} f(v),\ dv| \leq \sqrt{2}N^2\Delta\bar{f}|[r]|),$$
$$\text{and } d^{(1,\ldots,N)} \geq 1 - p_{soph}.$$

I can clearly take δ sufficiently small so that any point in $c^*(P')$ is within ϵ of a point in $c^*(P)$, for any $\epsilon > 0$.

□

Lemma A.4. *If $N = 3$, the equilibrium probabilities of admission, choice density vectors, and expected utilities of all students are unique.*

Proof. I restrict to equilibria where students submit $r = (2)$ if they are indifferent between submitting $r = 2$ and $r = (1, 2)$. This restriction does not impact the theorem because measure 0 students are indifferent. The restriction implies that sophisticated students submit the rank order

list $r = (2)$ if and only if

$$v_2 \cdot (p_2^{(2)}(c, \lambda) - p_2^{(1,2)}(c, \lambda)) - p_1^{(1,2)}(c, \lambda) \geq 0. \tag{A.2}$$

Equation A.2 implies that any equilibrium has a cutoff valuation, $\bar{v}_2$, such that all sophisticated students with $v_2 \geq \bar{v}_2$ submit $r = 2$ and all sophisticated students with $v_2 < \bar{v}_2$ submit $r = (1, 2)$. I plug in for the probabilities of admission in Equation A.2 to solve for $\bar{v}_2$ as a function of $c^{(1,2)}$:

$$\bar{v}_2 = \frac{q_1}{q_1 + \lambda_2 c^{(1,2)}}. \tag{A.3}$$

I now prove the lemma by contradiction. Suppose there are two different equilibria, σ and σ' with corresponding cutoff valuations $\bar{v}_2$ and $\bar{v}_2'$, such that $\bar{v}_2 < \bar{v}_2'$. If σ and σ' are, in fact, different equilibria, then there is a positive measure of students with valuations between $\bar{v}_2$ and $\bar{v}_2'$, which implies that $c^{(1,2)}(\sigma) < c^{(1,2)}(\sigma')$. Examining Equation A.3 shows that this implies that $\bar{v}_2 > \bar{v}_2'$ and the lemma is proven by contradiction. □

Lemma A.5. *All students with strictly monotonically increasing utility vectors submit the rank order list $r = (1, 2, \ldots, N)$ in any equilibrium of deferred acceptance.*

Proof. I prove the lemma by induction. The induction hypothesis is that if all students with strictly monotonically increasing utility vectors include schools 1 through n in their rank order list, then they include schools 1 through $n + 1$. The base case is trivial. All students with $v_1 > v_2$ include school 1 in their rank order list because doing so does not decrease their probability of admission at subsequent schools conditional on having a lottery number less than $\bar{L}_1$ and they strictly prefer school 1 to subsequent schools.

Now suppose all students with monotonically increasing utility vectors include schools 1 through n in their rank order list. There are measure 1 such students, so the probability of admission to one of those schools is $\sum_1^n q_s < 1$. Admission to one of those schools is not certain, and ranking school

$n+1$ next yields a positive probability of admission at school $n+1$ because measure 1 students included schools 1 through n in their ranking, and all students rejected at schools 1 through n have the same distribution of lottery numbers. School $n+1$ is strictly preferred to subsequent schools, and ranking it ahead of subsequent schools does not reduce the probability of admission at subsequent schools, conditional on having a lottery number less than $\bar{L}_{n+1}$. All students, therefore, include School $n+1$ in their ranking.

□

Lemma A.6. *All schools other than the outside option are at full capacity in any equilibrium of the Taiwan mechanism.*

Proof. Fix an equilibrium strategy, σ, and suppose School s is not at full capacity under σ. Then σ instructs a positive measure of students who have a positive probability of admission at some $t > s$ to exclude s from their rank order list. Including s in their rank order list is a profitable deviation for these students because it guarantees admission to s conditional on rejection at all schools ranked ahead of s, so σ is not an equilibrium strategy. The lemma is proven by contradiction.

□

Lemma A.7. *The derivative of c with respect to λ exists at $\lambda = 0$.*

Proof. Define a function, G, where $G(c,\lambda) = c^*(c,\lambda) - c$ and c^* is defined as in the proof of Theorem 1.1. In general, c^* is a correspondence, not a function, but the characterization of c^* in the proof of Lemma A.2 shows that $c^*(c,\lambda)$ is a singleton whenever the equivalence class $[r]$ is a singleton for all r. This is true when λ is sufficiently close to 0 and measure 1 students submit the ranking $r = (1, 2, \ldots, N)$. Both conditions hold everywhere in this proof, so the characterization of c^* simplifies to

$$c^{r^*}(c,\lambda) = (1 - p_{soph})\mathbb{I}(r = (1,\ldots,N)) + p_{soph}\int_{V^r(c,\lambda)} f(v).$$

I use the implicit function theorem to prove the lemma. This requires demonstrating that:

1. $G(c, \lambda)$ is C1

2. The Jacobian of G with respect to c does not equal 0.

I use the main result from Müller et al. (2011) and the chain rule, to compute the derivative of $c^{r^*}(c, \lambda)$ with respect to λ for a fixed r:

$$\frac{\partial c^{r^*}(c,\lambda)}{\partial \lambda} = p_{soph} \sum_{s,r'} \frac{\partial (P_s^r(c,\lambda) - P_s^{r'}(c,\lambda))}{\partial \lambda} \frac{1}{\|P^r(c,\lambda) - P^{r'}(c,\lambda)\|} \int_{\mathscr{F}_i} v_s f(v) \mathrm{d}\mathscr{F}_i,$$

where

$$\mathscr{F}_i = \Big\{ \mathbf{v} \in \mathbb{V}^N \mid v \cdot (P^r(c,\lambda) - P^{r'}(c,\lambda)) = 0,$$
$$v \cdot (P^r(c,\lambda) - P^{r'}(c,\lambda)) \geq 0 \text{ for all } r'' \in \mathcal{R}, \text{ and } 1 \geq v_1 \geq \ldots \geq v_N = 0 \Big\}.$$

and the integration is with respect to the $(n-1)$ dimensional affine variety that contains $\mathscr{F}_i$.

The continuity of the expression follows immediately from the continuity of f and the fact that P is a linear function of λ. The derivative of G with respect to λ is the vector of the above expression for each r, so it is also continuous.

Next, I consider the Jacobian of G with respect to c. Lemma A.5 demonstrates that if $\lambda = 0$, measure 1 students submit the rank order list $(1, 2, \ldots, N)$ regardless of the other students' choices, so $\frac{\partial c^*(c,0)}{c} = 0$. The Jacobian of G with respect to c is, therefore, $-\mathbb{I}$. Both conditions of the implicit function theorem follow immediately, which concludes the proof. □

Lemma A.8. *If f has full support, then $\sum_{r:s\notin r} \partial c^r > 0$ for some s at $\lambda = 0$.*

All students submit $r = (1, 2, \ldots, N)$ when $\lambda = 0$, so it is sufficient to show that the derivative of $c^{(1,2,\ldots,N)}$ with respect to λ is strictly negative at $\lambda = 0$. Applying the implicit function theorem

and the results in Lemma A.7 shows that the derivative is

$$p_{soph} \sum_{s,r'} \frac{\partial(P_s^r(c,\lambda) - P_s^{r'}(c,\lambda))}{\partial\lambda} \frac{1}{\|P^r(c,\lambda) - P^{r'}(c,\lambda)\|} \int_{\mathscr{F}_i} v_s f(v) \mathrm{d}\mathscr{F}_i, \tag{A.4}$$

where $r = (1, 2, \ldots, N)$. The distribution of utility vectors has full support, so $\int_{\mathscr{F}_i} v_s f(v)$ is non-zero for all r', s. All that remains is to compute $\frac{\partial(P_s^r(c,\lambda) - P_s^{r'}(c,\lambda))}{\partial\lambda}$. First, note that $\frac{\partial P_s^r(c,\lambda)}{\partial\lambda} = 0$ because measure 1 students submit their true preferences, so the change in the deduction rule impacts them all equally (see the proof of Theorem 1.4).

Recall from Equation 1.6 that $-P_s^{r'}(c,\lambda) = -(1 - \sum_{t<s} p_t^r - \bar{L}_s - \lambda_{r'(s)})^+$. Note that $\frac{\partial \bar{L}_s}{\partial\lambda} = -\lambda_s$ because measure 1 students submit their true preferences (Equation 1.5). It follows that $\frac{\partial P_s^{r'}(c,\lambda)}{\partial\lambda} = \frac{\partial \sum_{t<s} p_t^r}{\partial\lambda} - \partial\lambda_s + \partial\lambda_{r'(s)}$. Observe that $-\partial\lambda_s + \partial\lambda_{r'(s)}$ is weakly negative because $s \geq r'(s)$ and λ is weakly monotonically increasing. It also must be strictly negative for some r',s because λ has increased from 0. The remaining term, $\frac{\partial \sum_{t<s} p_t^r}{\partial\lambda}$ may be positive, but that implies that $\frac{\partial(P_s^r(c,\lambda) - P_s^{r'}(c,\lambda))}{\partial\lambda}$ was negative for an earlier s, and that term is multiplied by a weakly larger term because v_s is monotonically decreasing. The entire term is, therefore, negative, which completes the proof.

Lemma A.9. *If $N = 3$ schools, the derivative of c with respect to λ_2 exists.*

Proof. Simple algebra using the probabilities from Equation (1.6) shows that students prefer to rank School 2 first if $v_2 > \frac{q_1}{(1-c^{(2)})\lambda_2 + q_1}$. It follows that $c^{(2)^*}(c^{(2)}, \lambda_2) = p_{soph}\left(1 - F(\frac{q_1}{(1-c^{(2)})\lambda_2 + q_1})\right)$. The game is in equilibrium if $G(c^{(2)}, \lambda) = 0$, where

$$G(c^{(2)}, \lambda_2) = c^{(2)^*}(c^{(2)}, \lambda_2) - c = p_{soph}\left(1 - F(\frac{q_1}{(1 - c^{(2)})\lambda_2 + q_1})\right) - c^{(2)}.$$

The implicit function theorem states that if $G(C, \lambda_2)$ is C1 and $G_{c^{(2)}}(c^{(2)}, \lambda_2) \neq 0$, then $c^{(2)}$ is

differentiable with respect to λ_2. I take the derivative of G with respect to λ_2:

$$\frac{\partial}{\partial \lambda_2}\left(-F\left(\frac{q_1}{(1-c^{(2)})\lambda_2+q_1}\right)-c\right)=\frac{(1-c^{(2)})q_1 f\left(\frac{q_1}{(1-c^{(2)})\lambda_2+q_1}\right)}{\left((1-c^{(2)})\lambda_2+q_1\right)^2}$$

I take the derivative of G w.r.t c:

$$\frac{\partial}{\partial c^{(2)}}\left(-F\left(\frac{q_1}{(1-c^{(2)})\lambda_2+q_1}\right)-c^{(2)}\right)=-\frac{\lambda_2 q_1 f\left(\frac{q_1}{(1-c^{(2)})\lambda_2+q_1}\right)}{\left((1-c^{(2)})\lambda_2+q_1\right)^2}-1$$

It is apparent from the equations that the continuity of f implies G is C1. Furthermore, $G_c(c^{(2)},\lambda_2)<0$ because $f(\cdot)\geq 0$ and $c^{(2)}<1$. □

Lemma A.10. *Under a constant deduction rule, $\Lambda=(0,\lambda,\lambda,\ldots)$ the student always has a best response of the form $r=(s_i,s_j,s_k,s_l,\ldots)$, where $v(s_j)>v(s_k)>v(s_l)>\ldots$ In words, the student chooses the top school strategically, but subsequently reveals their true preferences.*

Proof. Suppose otherwise. Then there exists r, such that the student prefers s_j to s_k, neither ranks s_j nor s_k first, ranks s_k immediately ahead of s_j, and $u(r)>u(r'')$ for any r'' that truthfully reveals preferences after the first choice.

Consider r' identical to r, except that the positions of s_j and s_k are flipped. If the student is admitted to a school ranked ahead of s_j and s_k under r, then they will be admitted to the same school under r', so utility is unchanged. If they are admitted to s_j under r, then they will also be admitted to s_j under r' because s_j is ranked higher under r'. If they are admitted to s_k under r, then their lottery number, $l-\lambda>\bar{L_k}$. They will, therefore, be admitted to s_k under r' when $l-\lambda<\bar{L_j}$, and s_j otherwise. Because the student prefers s_j this is a weak increase in utility. Finally, if the student is admitted to a school, s_l, ranked after s_j and s_k, under r, then their lottery number is $l-\lambda>\bar{L_l}$, which implies they will also be admitted under r'.

If r' is of the form $r=(s_i,s_j,s_k,s_l,\ldots)$, where $v(s_j)>v(s_k)>v(s_l)>\ldots$, then the lemma is proven by contradiction. If not, repeat the process described in the previous paragraph. The rank

order list is finite, so this will eventually conclude with a rank order list of the appropriate form that is weakly preferred to r.

□

A.7. Equation for Empirical P

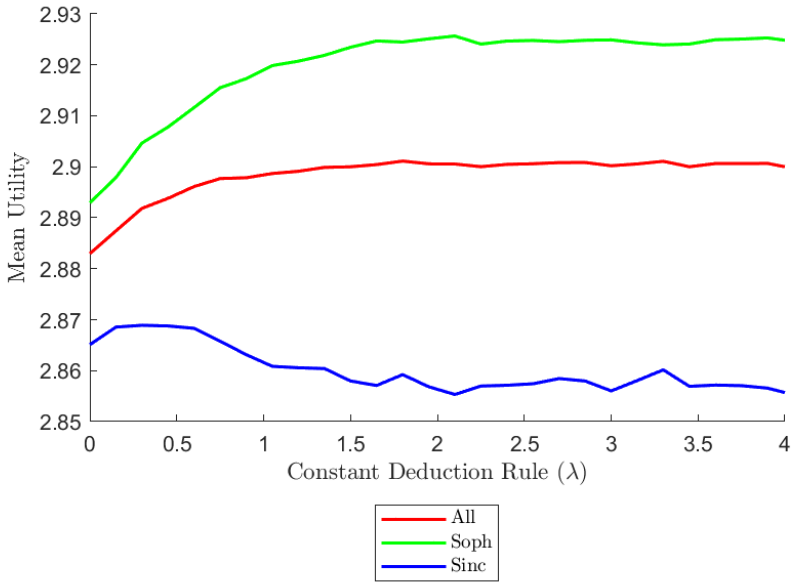
This section derives the equation $P(\bar{L})$ used in the empirical section. Let $p_s^{r,a(s)}$ denote the probability of admission to school s for a student who submitted rank order list r, and has priority $a(s)$ at s. Conditional on rejection at earlier ranked schools, the student's priority guarantees them admission to s if $a(s) - \lambda_{r(s)} > \bar{L}$. In this case, $p_s^{r,a(s)} = 1 - \sum_{t<s} p_t^r$. If $a(s) + 1 - \lambda_{r(s)} < \bar{L}$, the student cannot be admitted even with the best lottery number, so, $p_s^{r,a(s)} = 0$. Admission depends on the lottery number when $a(s) - \lambda_{r(s)} > \bar{L} > a(s) - \lambda_{r(s)}$. The probability of having a sufficiently high lottery number, conditional on rejection at earlier schools is $1 + a(s) - \sum_{t<s} p_t^r - \lambda_{r(s)} - \bar{L}$. The following equation summarizes the results.

$$p_s^{r,a(s)}(\bar{L_s}) = \begin{cases} (a(s) + 1 - \sum_{t<s} p_t^r - \bar{L}_s - \lambda_{r(s)})^+ & \text{if } \bar{L}_s > a(s) - \lambda_{r(s)}; \\ 1 - \sum_{t<s} p_t^r & \text{otherwise,} \end{cases} \tag{A.5}$$

where $(\cdot)^+$ indicates $max(\cdot, 0)$.

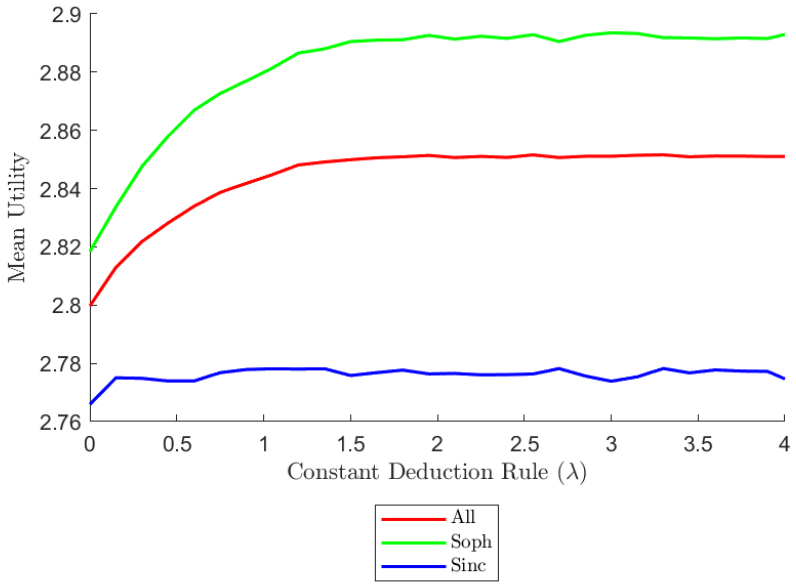
A.8. Additional Empirical Results

Figure A.1: Estimated Utility in 2004 for Constant Deduction Rules



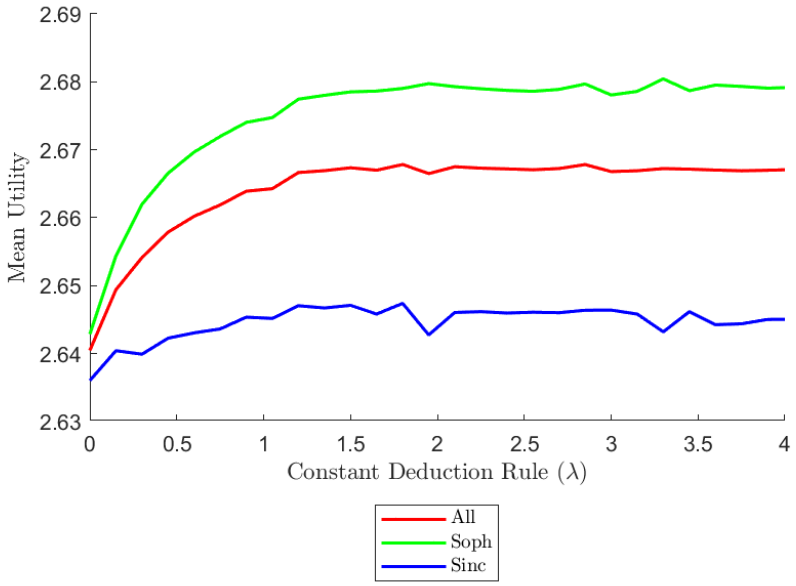
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.2: Estimated Utility in 2005 for Constant Deduction Rules



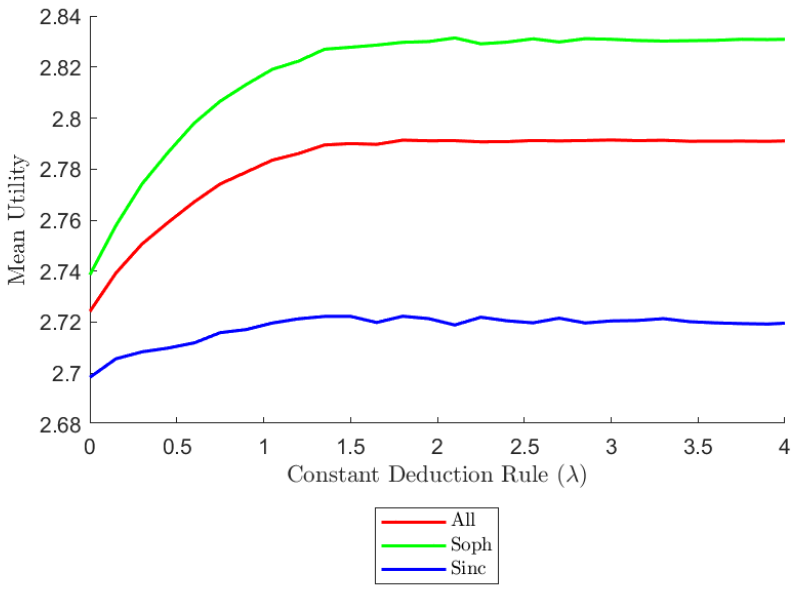
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.3: Estimated Utility in 2006 for Constant Deduction Rules



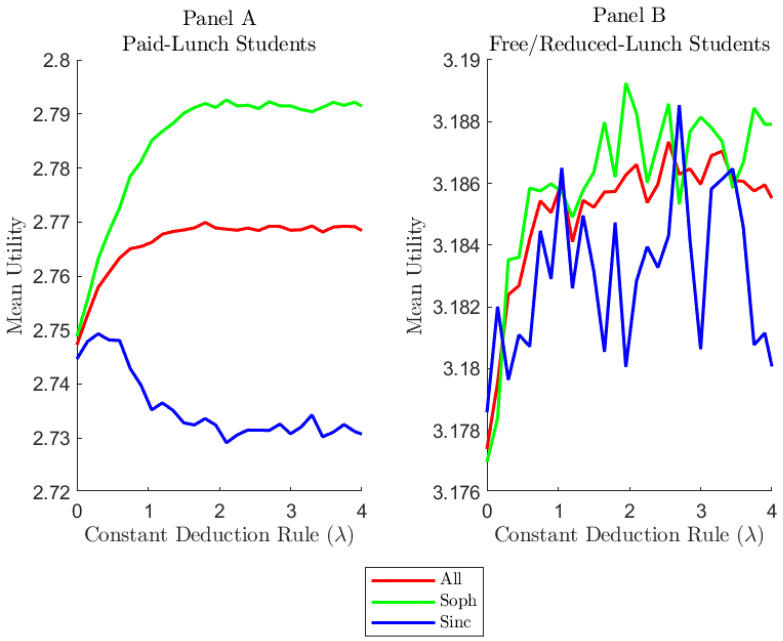
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.4: Estimated Utility in 2008 for Constant Deduction Rules



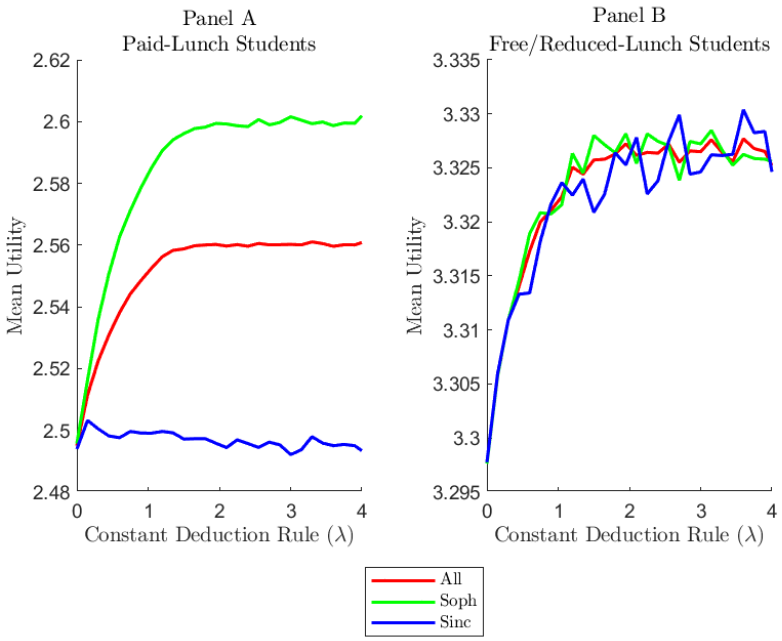
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. The capacity data for Tobin was nonsensical in 2008, so I manually entered a total capacity of 20 seats, and free/reduced-lunch and paid-lunch capacities of 10 seats each.

Figure A.5: Estimated Heterogeneous Utility in 2004 for Constant Deduction Rules



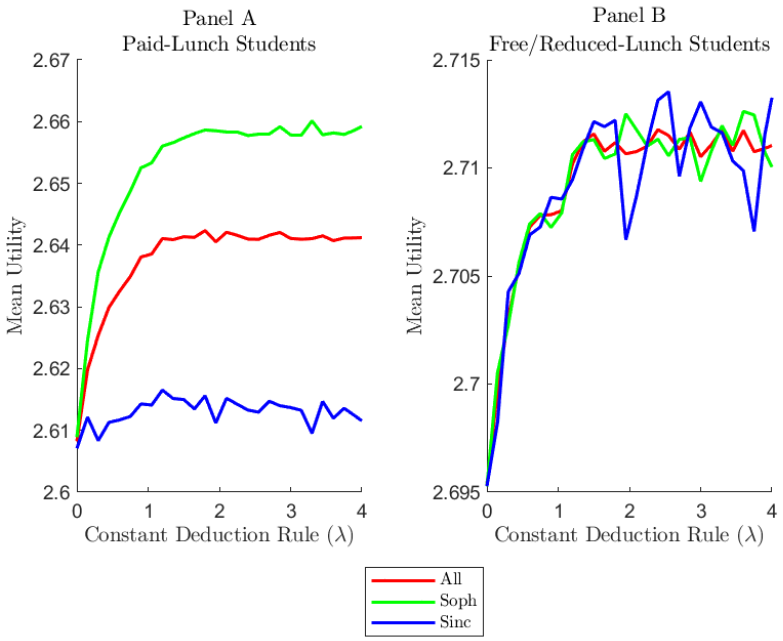
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. Panel A plots paid-lunch students' utility and panel B plots free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.6: Estimated Heterogeneous Utility in 2005 for Constant Deduction Rules



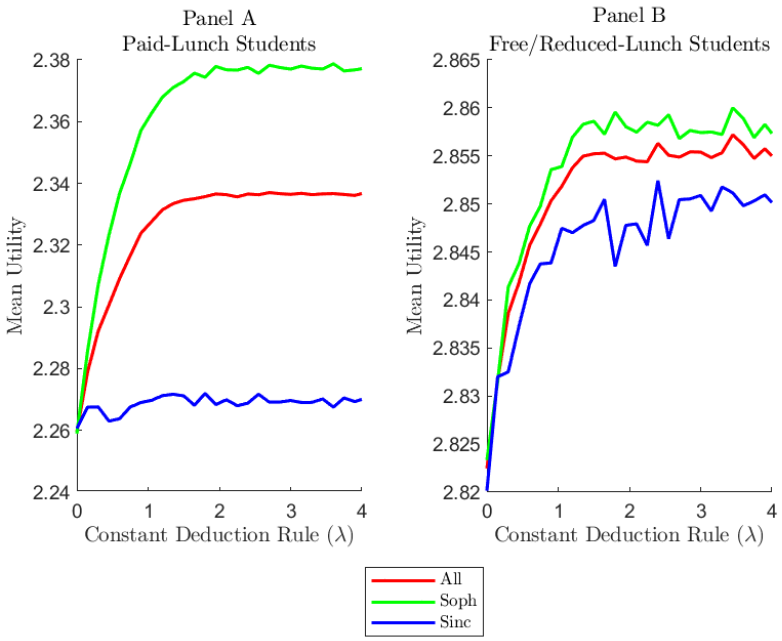
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. Panel A plots paid-lunch students' utility and panel B plots free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.7: Estimated Heterogeneous Utility in 2006 for Constant Deduction Rules



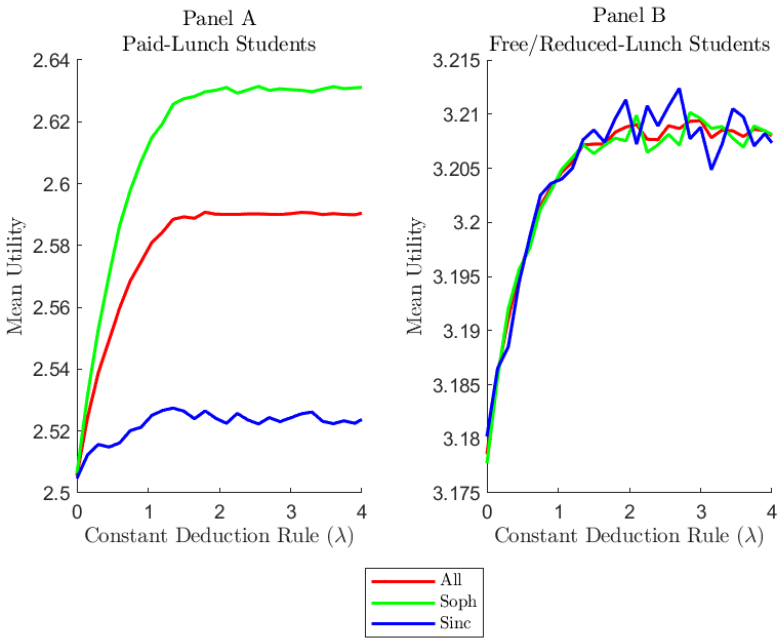
Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. Panel A plots paid-lunch students' utility and panel B plots free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.8: Estimated Heterogeneous Utility in 2007 for Constant Deduction Rules



Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. Panel A plots paid-lunch students' utility and panel B plots free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4.

Figure A.9: Estimated Heterogeneous Utility in 2008 for Constant Deduction Rules



Notes: This figure uses simulations of Cambridge students' utility to plot sincere students', sophisticated students', and all students' estimated welfare as a function of a constant deduction parameter, λ. Panel A plots paid-lunch students' utility and panel B plots free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. The capacity data for Tobin was nonsensical in 2008, so I manually entered a total capacity of 20 seats, and free/reduced-lunch and paid-lunch capacities of 10 seats each.

Table A.4: Estimated Utility in 2004 for Variable Deduction Rules

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.88	2.88	2.88	2.88	2.88	2.88
	$\lambda_2 = .4$	2.89	2.89	2.89	2.89	2.89	2.89
	$\lambda_2 = .8$	2.89	2.89	2.89	2.89	2.89	2.89
	$\lambda_2 = 1.2$	2.89	2.89	2.89	2.89	2.89	2.89
	$\lambda_2 = 1.6$	2.89	2.89	2.89	2.89	2.89	2.89
	$\lambda_2 = 2$	2.89	2.89	2.89	2.89	2.89	2.89
Sincere	$\lambda_2 = 0$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2 = .4$	2.85	2.84	2.84	2.84	2.84	2.84
	$\lambda_2 = .8$	2.84	2.84	2.83	2.83	2.83	2.83
	$\lambda_2 = 1.2$	2.83	2.83	2.83	2.83	2.83	2.83
	$\lambda_2 = 1.6$	2.83	2.83	2.83	2.83	2.83	2.83
	$\lambda_2 = 2$	2.83	2.83	2.83	2.83	2.83	2.83
Sophisticated	$\lambda_2 = 0$	2.90	2.90	2.90	2.90	2.90	2.90
	$\lambda_2 = .4$	2.91	2.91	2.91	2.91	2.91	2.91
	$\lambda_2 = .8$	2.91	2.92	2.92	2.92	2.92	2.92
	$\lambda_2 = 1.2$	2.92	2.92	2.92	2.92	2.92	2.92
	$\lambda_2 = 1.6$	2.92	2.92	2.92	2.92	2.92	2.92
	$\lambda_2 = 2$	2.92	2.92	2.92	2.92	2.92	2.92

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.5: Estimated Utility in 2005 for Variable Deduction Rules

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.81	2.82	2.82	2.82	2.82	2.82
	$\lambda_2 = .4$	2.83	2.83	2.83	2.84	2.83	2.84
	$\lambda_2 = .8$	2.84	2.84	NA	2.84	2.84	2.84
	$\lambda_2 = 1.2$	2.84	2.84	NA	2.84	2.84	2.84
	$\lambda_2 = 1.6$	2.84	2.84	NA	2.84	2.84	2.84
	$\lambda_2 = 2$	2.84	2.84	2.84	2.84	2.84	2.84
Sincere	$\lambda_2 = 0$	2.76	2.76	2.76	2.77	2.77	2.77
	$\lambda_2 = .4$	2.76	2.76	2.76	2.76	2.76	2.76
	$\lambda_2 = .8$	2.76	2.76	NA	2.76	2.76	2.76
	$\lambda_2 = 1.2$	2.76	2.76	NA	2.76	2.76	2.76
	$\lambda_2 = 1.6$	2.76	2.76	NA	2.76	2.76	2.76
	$\lambda_2 = 2$	2.76	2.76	2.76	2.76	2.76	2.75
Sophisticated	$\lambda_2 = 0$	2.83	2.85	2.85	2.85	2.86	2.85
	$\lambda_2 = .4$	2.86	2.87	2.87	2.88	2.88	2.88
	$\lambda_2 = .8$	2.88	2.88	NA	2.88	2.88	2.88
	$\lambda_2 = 1.2$	2.88	2.89	NA	2.89	2.89	2.89
	$\lambda_2 = 1.6$	2.89	2.89	NA	2.89	2.89	2.89
	$\lambda_2 = 2$	2.89	2.89	2.89	2.89	2.89	2.89

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.6: Estimated Utility in 2006 for Variable Deduction Rules

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.64	2.65	2.65	2.65	2.65	2.65
	$\lambda_2 = .4$	2.65	2.65	2.65	2.65	2.65	2.65
	$\lambda_2 = .8$	2.66	2.66	2.66	2.66	2.66	2.66
	$\lambda_2 = 1.2$	2.66	2.66	2.66	2.66	2.66	2.66
	$\lambda_2 = 1.6$	2.66	2.66	2.66	2.66	2.66	2.66
	$\lambda_2 = 2$	2.66	2.66	2.66	2.66	2.66	2.66
Sincere	$\lambda_2 = 0$	2.63	2.63	2.63	2.63	2.63	2.63
	$\lambda_2 = .4$	2.63	2.62	2.62	2.62	2.62	2.62
	$\lambda_2 = .8$	2.63	2.62	2.62	2.62	2.62	2.62
	$\lambda_2 = 1.2$	2.63	2.62	2.62	2.62	2.62	2.62
	$\lambda_2 = 1.6$	2.63	2.62	2.62	2.62	2.62	2.62
	$\lambda_2 = 2$	2.63	2.62	2.62	2.62	2.62	2.62
Sophisticated	$\lambda_2 = 0$	2.65	2.65	2.66	2.66	2.66	2.66
	$\lambda_2 = .4$	2.67	2.67	2.67	2.67	2.67	2.67
	$\lambda_2 = .8$	2.67	2.68	2.68	2.68	2.68	2.68
	$\lambda_2 = 1.2$	2.68	2.68	2.68	2.68	2.68	2.68
	$\lambda_2 = 1.6$	2.68	2.68	2.68	2.68	2.68	2.68
	$\lambda_2 = 2$	2.68	2.68	2.68	2.68	2.68	2.68

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.7: Estimated Utility in 2008 for Variable Deduction Rules

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.73	2.74	2.75	2.75	2.75	2.75
	$\lambda_2 = .4$	2.76	2.77	2.77	2.77	2.77	2.77
	$\lambda_2 = .8$	2.77	2.78	2.78	2.78	2.78	2.78
	$\lambda_2 = 1.2$	2.78	2.78	2.78	2.78	2.78	2.78
	$\lambda_2 = 1.6$	2.78	2.78	2.78	2.78	2.78	2.78
	$\lambda_2 = 2$	2.78	2.78	2.78	2.78	2.78	2.78
Sincere	$\lambda_2 = 0$	2.69	2.69	2.70	2.70	2.70	2.70
	$\lambda_2 = .4$	2.69	2.69	2.70	2.70	2.70	2.70
	$\lambda_2 = .8$	2.70	2.70	2.70	2.70	2.70	2.70
	$\lambda_2 = 1.2$	2.70	2.70	2.70	2.70	2.70	2.70
	$\lambda_2 = 1.6$	2.70	2.70	2.70	2.70	2.70	2.70
	$\lambda_2 = 2$	2.70	2.70	2.70	2.70	2.70	2.70
Sophisticated	$\lambda_2 = 0$	2.76	2.77	2.78	2.78	2.78	2.78
	$\lambda_2 = .4$	2.80	2.81	2.81	2.81	2.81	2.81
	$\lambda_2 = .8$	2.81	2.82	2.82	2.82	2.82	2.82
	$\lambda_2 = 1.2$	2.82	2.82	2.83	2.83	2.83	2.83
	$\lambda_2 = 1.6$	2.83	2.83	2.83	2.83	2.83	2.83
	$\lambda_2 = 2$	2.83	2.83	2.83	2.83	2.83	2.83

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule. The capacity data for Tobin was nonsensical in 2008, so I manually entered a total capacity of 20 seats, and free/reduced-lunch and paid-lunch capacities of 10 seats each.

Table A.8: Estimated Heterogeneous Utility in 2004 for Variable Deduction Rules

Panel A: Paid-Lunch Students

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.74	2.74	2.74	2.74	2.74	2.74
	$\lambda_2 = .4$	2.75	2.75	2.75	2.75	2.75	2.75
	$\lambda_2 = .8$	2.75	2.75	2.75	2.75	2.75	2.75
	$\lambda_2 = 1.2$	2.75	2.75	2.75	2.75	2.75	2.75
	$\lambda_2 = 1.6$	2.75	2.75	2.75	2.75	2.75	2.75
	$\lambda_2 = 2$	2.75	2.75	2.75	2.75	2.75	2.75
Sincere	$\lambda_2 = 0$	2.72	2.72	2.73	2.72	2.72	2.72
	$\lambda_2 = .4$	2.72	2.72	2.71	2.71	2.71	2.71
	$\lambda_2 = .8$	2.71	2.70	2.70	2.70	2.70	2.70
	$\lambda_2 = 1.2$	2.70	2.70	2.70	2.70	2.70	2.69
	$\lambda_2 = 1.6$	2.70	2.69	2.69	2.69	2.69	2.69
	$\lambda_2 = 2$	2.69	2.69	2.69	2.69	2.69	2.69
Sophisticated	$\lambda_2 = 0$	2.75	2.76	2.76	2.76	2.76	2.76
	$\lambda_2 = .4$	2.77	2.77	2.77	2.77	2.77	2.77
	$\lambda_2 = .8$	2.78	2.78	2.78	2.78	2.78	2.78
	$\lambda_2 = 1.2$	2.79	2.79	2.79	2.79	2.79	2.79
	$\lambda_2 = 1.6$	2.79	2.79	2.79	2.79	2.79	2.79
	$\lambda_2 = 2$	2.79	2.79	2.79	2.79	2.79	2.79

Panel B: Free/Reduced-Lunch Students

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = .4$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = .8$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 1.2$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 1.6$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 2$	3.18	3.18	3.18	3.18	3.18	3.18
Sincere	$\lambda_2 = 0$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = .4$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = .8$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 1.2$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 1.6$	3.18	3.18	3.17	3.18	3.18	3.18
	$\lambda_2 = 2$	3.18	3.17	3.18	3.17	3.18	3.17
Sophisticated	$\lambda_2 = 0$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = .4$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = .8$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 1.2$	3.18	3.18	3.18	3.18	3.18	3.18
	$\lambda_2 = 1.6$	3.18	3.18	3.19	3.18	3.19	3.18
	$\lambda_2 = 2$	3.18	3.19	3.19	3.19	3.18	3.18

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. Panel A reports paid-lunch students' utility and panel B reports free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.9: Estimated Heterogeneous Utility in 2005 for Variable Deduction Rules

Panel A: Paid-Lunch Students

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	2.50	2.52	2.52	2.52	2.53	2.53
	$\lambda_2 = .4$	2.53	2.54	2.54	2.54	2.54	2.54
	$\lambda_2 = .8$	2.54	2.54	NA	2.54	2.54	2.54
	$\lambda_2 = 1.2$	2.55	2.55	NA	2.55	2.55	2.55
	$\lambda_2 = 1.6$	2.55	2.55	NA	2.55	2.55	2.55
	$\lambda_2 = 2$	2.55	2.55	2.55	2.55	2.55	2.55
Sincere	$\lambda_2 = 0$	2.48	2.48	2.49	2.49	2.49	2.49
	$\lambda_2 = .4$	2.48	2.48	2.48	2.47	2.47	2.47
	$\lambda_2 = .8$	2.48	2.47	NA	2.47	2.47	2.47
	$\lambda_2 = 1.2$	2.48	2.47	NA	2.47	2.47	2.47
	$\lambda_2 = 1.6$	2.47	2.47	NA	2.47	2.47	2.47
	$\lambda_2 = 2$	2.47	2.47	2.47	2.47	2.47	2.46
Sophisticated	$\lambda_2 = 0$	2.52	2.54	2.54	2.55	2.55	2.55
	$\lambda_2 = .4$	2.56	2.57	2.58	2.58	2.58	2.58
	$\lambda_2 = .8$	2.58	2.59	NA	2.59	2.59	2.59
	$\lambda_2 = 1.2$	2.59	2.59	NA	2.59	2.59	2.59
	$\lambda_2 = 1.6$	2.59	2.59	NA	2.60	2.59	2.59
	$\lambda_2 = 2$	2.59	2.60	2.59	2.60	2.60	2.60

Panel B: Free/Reduced-Lunch Students

		$\lambda_3 - \lambda_2 = 0$	$\lambda_3 - \lambda_2 = .4$	$\lambda_3 - \lambda_2 = .8$	$\lambda_3 - \lambda_2 = 1.2$	$\lambda_3 - \lambda_2 = 1.6$	$\lambda_3 - \lambda_2 = 2$
All	$\lambda_2 = 0$	3.30	3.31	3.31	3.31	3.31	3.31
	$\lambda_2 = .4$	3.32	3.32	3.32	3.32	3.32	3.32
	$\lambda_2 = .8$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 1.2$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 1.6$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 2$	3.32	3.32	3.32	3.32	3.32	3.32
Sincere	$\lambda_2 = 0$	3.30	3.31	3.31	3.31	3.31	3.31
	$\lambda_2 = .4$	3.31	3.31	3.32	3.32	3.32	3.32
	$\lambda_2 = .8$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 1.2$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 1.6$	3.32	3.32	NA	3.33	3.32	3.32
	$\lambda_2 = 2$	3.32	3.32	3.32	3.32	3.32	3.32
Sophisticated	$\lambda_2 = 0$	3.30	3.31	3.31	3.31	3.31	3.31
	$\lambda_2 = .4$	3.32	3.32	3.32	3.32	3.32	3.32
	$\lambda_2 = .8$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 1.2$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 1.6$	3.32	3.32	NA	3.32	3.32	3.32
	$\lambda_2 = 2$	3.32	3.32	3.32	3.32	3.32	3.32

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. Panel A reports paid-lunch students' utility and panel B reports free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.10: Estimated Heterogeneous Utility in 2006 for Variable Deduction Rules

Panel A: Paid-Lunch Students

		$\lambda_3-\lambda_2=0$	$\lambda_3-\lambda_2=.4$	$\lambda_3-\lambda_2=.8$	$\lambda_3-\lambda_2=1.2$	$\lambda_3-\lambda_2=1.6$	$\lambda_3-\lambda_2=2$
All	$\lambda_2=0$	2.61	2.61	2.61	2.61	2.62	2.62
	$\lambda_2=.4$	2.62	2.62	2.62	2.62	2.62	2.62
	$\lambda_2=.8$	2.63	2.63	2.63	2.63	2.63	2.63
	$\lambda_2=1.2$	2.63	2.63	2.63	2.63	2.63	2.63
	$\lambda_2=1.6$	2.63	2.63	2.63	2.63	2.63	2.63
	$\lambda_2=2$	2.63	2.63	2.63	2.63	2.63	2.63
Sincere	$\lambda_2=0$	2.59	2.59	2.59	2.59	2.59	2.59
	$\lambda_2=.4$	2.59	2.58	2.58	2.58	2.58	2.58
	$\lambda_2=.8$	2.59	2.58	2.58	2.58	2.58	2.58
	$\lambda_2=1.2$	2.59	2.58	2.58	2.58	2.58	2.58
	$\lambda_2=1.6$	2.59	2.58	2.58	2.58	2.58	2.58
	$\lambda_2=2$	2.58	2.58	2.58	2.58	2.58	2.58
Sophisticated	$\lambda_2=0$	2.62	2.63	2.63	2.63	2.63	2.63
	$\lambda_2=.4$	2.64	2.65	2.65	2.65	2.65	2.65
	$\lambda_2=.8$	2.65	2.65	2.65	2.65	2.65	2.65
	$\lambda_2=1.2$	2.66	2.66	2.66	2.66	2.66	2.66
	$\lambda_2=1.6$	2.66	2.66	2.66	2.66	2.66	2.66
	$\lambda_2=2$	2.66	2.66	2.66	2.66	2.66	2.66

Panel B: Free/Reduced-Lunch Students

		$\lambda_3-\lambda_2=0$	$\lambda_3-\lambda_2=.4$	$\lambda_3-\lambda_2=.8$	$\lambda_3-\lambda_2=1.2$	$\lambda_3-\lambda_2=1.6$	$\lambda_3-\lambda_2=2$
All	$\lambda_2=0$	2.70	2.70	2.70	2.70	2.70	2.70
	$\lambda_2=.4$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=.8$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=1.2$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=1.6$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=2$	2.71	2.71	2.71	2.71	2.71	2.71
Sincere	$\lambda_2=0$	2.70	2.70	2.70	2.70	2.70	2.70
	$\lambda_2=.4$	2.71	2.70	2.71	2.71	2.71	2.71
	$\lambda_2=.8$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=1.2$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=1.6$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=2$	2.71	2.71	2.71	2.71	2.71	2.71
Sophisticated	$\lambda_2=0$	2.70	2.70	2.70	2.70	2.70	2.70
	$\lambda_2=.4$	2.70	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=.8$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=1.2$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=1.6$	2.71	2.71	2.71	2.71	2.71	2.71
	$\lambda_2=2$	2.71	2.71	2.71	2.71	2.71	2.71

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. Panel A reports paid-lunch students' utility and panel B reports free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.11: Estimated Heterogeneous Utility in 2007 for Variable Deduction Rules

Panel A: Paid-Lunch Students

		$\lambda_3-\lambda_2=0$	$\lambda_3-\lambda_2=.4$	$\lambda_3-\lambda_2=.8$	$\lambda_3-\lambda_2=1.2$	$\lambda_3-\lambda_2=1.6$	$\lambda_3-\lambda_2=2$
All	$\lambda_2=0$	2.27	2.28	2.29	2.29	2.29	2.29
	$\lambda_2=.4$	2.30	2.31	2.31	2.31	2.31	2.31
	$\lambda_2=.8$	2.32	2.32	2.32	2.32	2.32	2.32
	$\lambda_2=1.2$	2.32	2.32	2.32	2.32	2.32	2.32
	$\lambda_2=1.6$	2.33	2.33	2.32	2.32	2.32	2.32
	$\lambda_2=2$	2.33	2.33	2.33	2.32	2.32	2.32
Sincere	$\lambda_2=0$	2.25	2.25	2.26	2.26	2.26	2.26
	$\lambda_2=.4$	2.25	2.25	2.25	2.25	2.25	2.25
	$\lambda_2=.8$	2.25	2.25	2.24	2.24	2.24	2.24
	$\lambda_2=1.2$	2.25	2.25	2.24	2.24	2.24	2.24
	$\lambda_2=1.6$	2.25	2.25	2.24	2.24	2.24	2.24
	$\lambda_2=2$	2.25	2.25	2.24	2.24	2.24	2.24
Sophisticated	$\lambda_2=0$	2.28	2.30	2.31	2.31	2.31	2.31
	$\lambda_2=.4$	2.33	2.34	2.35	2.35	2.35	2.35
	$\lambda_2=.8$	2.35	2.36	2.37	2.37	2.37	2.37
	$\lambda_2=1.2$	2.37	2.37	2.37	2.37	2.37	2.37
	$\lambda_2=1.6$	2.37	2.37	2.37	2.38	2.38	2.38
	$\lambda_2=2$	2.38	2.37	2.38	2.38	2.38	2.38

Panel B: Free/Reduced-Lunch Students

		$\lambda_3-\lambda_2=0$	$\lambda_3-\lambda_2=.4$	$\lambda_3-\lambda_2=.8$	$\lambda_3-\lambda_2=1.2$	$\lambda_3-\lambda_2=1.6$	$\lambda_3-\lambda_2=2$
All	$\lambda_2=0$	2.83	2.84	2.84	2.84	2.84	2.84
	$\lambda_2=.4$	2.84	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=.8$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=1.2$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=1.6$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=2$	2.85	2.85	2.85	2.85	2.86	2.85
Sincere	$\lambda_2=0$	2.83	2.84	2.84	2.84	2.84	2.84
	$\lambda_2=.4$	2.84	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=.8$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=1.2$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=1.6$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=2$	2.85	2.85	2.85	2.85	2.85	2.85
Sophisticated	$\lambda_2=0$	2.83	2.84	2.84	2.84	2.84	2.84
	$\lambda_2=.4$	2.84	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=.8$	2.85	2.85	2.85	2.85	2.85	2.85
	$\lambda_2=1.2$	2.85	2.85	2.85	2.86	2.86	2.85
	$\lambda_2=1.6$	2.85	2.86	2.85	2.86	2.86	2.86
	$\lambda_2=2$	2.85	2.85	2.86	2.86	2.86	2.86

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. Panel A reports paid-lunch students' utility and panel B reports free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule.

Table A.12: Estimated Heterogeneous Utility in 2008 for Variable Deduction Rules

Panel A: Paid-Lunch Students

		$\lambda_3-\lambda_2=0$	$\lambda_3-\lambda_2=.4$	$\lambda_3-\lambda_2=.8$	$\lambda_3-\lambda_2=1.2$	$\lambda_3-\lambda_2=1.6$	$\lambda_3-\lambda_2=2$
All	$\lambda_2=0$	2.51	2.53	2.54	2.54	2.54	2.54
	$\lambda_2=.4$	2.55	2.56	2.56	2.56	2.56	2.56
	$\lambda_2=.8$	2.57	2.57	2.57	2.57	2.57	2.57
	$\lambda_2=1.2$	2.58	2.58	2.58	2.58	2.58	2.58
	$\lambda_2=1.6$	2.58	2.58	2.58	2.58	2.58	2.58
	$\lambda_2=2$	2.58	2.58	2.58	2.58	2.58	2.58
Sincere	$\lambda_2=0$	2.49	2.50	2.50	2.50	2.50	2.50
	$\lambda_2=.4$	2.49	2.49	2.49	2.49	2.49	2.49
	$\lambda_2=.8$	2.50	2.50	2.50	2.50	2.49	2.49
	$\lambda_2=1.2$	2.50	2.50	2.50	2.49	2.49	2.49
	$\lambda_2=1.6$	2.50	2.50	2.49	2.49	2.49	2.49
	$\lambda_2=2$	2.49	2.50	2.49	2.49	2.49	2.49
Sophisticated	$\lambda_2=0$	2.53	2.55	2.56	2.57	2.57	2.57
	$\lambda_2=.4$	2.58	2.60	2.60	2.60	2.60	2.60
	$\lambda_2=.8$	2.61	2.61	2.62	2.62	2.62	2.62
	$\lambda_2=1.2$	2.62	2.62	2.62	2.62	2.63	2.62
	$\lambda_2=1.6$	2.63	2.63	2.63	2.63	2.63	2.63
	$\lambda_2=2$	2.63	2.63	2.63	2.63	2.63	2.63

Panel B: Free/Reduced-Lunch Students

		$\lambda_3-\lambda_2=0$	$\lambda_3-\lambda_2=.4$	$\lambda_3-\lambda_2=.8$	$\lambda_3-\lambda_2=1.2$	$\lambda_3-\lambda_2=1.6$	$\lambda_3-\lambda_2=2$
All	$\lambda_2=0$	3.18	3.19	3.19	3.19	3.19	3.19
	$\lambda_2=.4$	3.20	3.20	3.20	3.20	3.20	3.20
	$\lambda_2=.8$	3.20	3.20	3.20	3.20	3.20	3.20
	$\lambda_2=1.2$	3.20	3.21	3.21	3.21	3.21	3.21
	$\lambda_2=1.6$	3.21	3.21	3.21	3.21	3.21	3.21
	$\lambda_2=2$	3.21	3.21	3.21	3.21	3.21	3.21
Sincere	$\lambda_2=0$	3.18	3.19	3.19	3.19	3.19	3.19
	$\lambda_2=.4$	3.20	3.20	3.20	3.20	3.20	3.20
	$\lambda_2=.8$	3.20	3.20	3.20	3.21	3.20	3.21
	$\lambda_2=1.2$	3.20	3.21	3.21	3.20	3.20	3.21
	$\lambda_2=1.6$	3.20	3.21	3.21	3.21	3.21	3.21
	$\lambda_2=2$	3.21	3.21	3.21	3.20	3.21	3.20
Sophisticated	$\lambda_2=0$	3.18	3.19	3.19	3.19	3.19	3.19
	$\lambda_2=.4$	3.20	3.20	3.20	3.20	3.20	3.20
	$\lambda_2=.8$	3.20	3.20	3.21	3.20	3.20	3.20
	$\lambda_2=1.2$	3.21	3.20	3.21	3.21	3.21	3.21
	$\lambda_2=1.6$	3.21	3.21	3.21	3.21	3.21	3.21
	$\lambda_2=2$	3.21	3.21	3.21	3.21	3.21	3.21

Notes: This table reports estimated utility for variable deduction rules from simulations of Cambridge students' utility. Panel A reports paid-lunch students' utility and panel B reports free/reduced-lunch students' utility. The priority structure, number of students, student demographics, and school capacities are given in the Cambridge data and summarized in Table 1.3. Utility for each student in the Cambridge data is drawn from the distribution specified in Table 1.4. Students must submit rank order lists with 3 schools or fewer. Entry is NA if the algorithm did not converge to an equilibrium for that deduction rule. The capacity data for Tobin was nonsensical in 2008, so I manually entered a total capacity of 20 seats, and free/reduced-lunch and paid-lunch capacities of 10 seats each.

Table A.13: Multiple Equilibria Robustness Test

Year	Initialization	Lunch Category	Max Difference	Mean Difference
2004	High	Free/Reduced	0.0090	0.0005
2004	Low	Free/Reduced	0.0099	0.0005
2004	High	Paid	0.0090	0.0003
2004	Low	Paid	0.0086	0.0003
2005	High	Free/Reduced	0.0095	0.0005
2005	Low	Free/Reduced	0.0099	0.0005
2005	High	Paid	0.0096	0.0003
2005	Low	Paid	0.0105	0.0003
2006	High	Free/Reduced	0.0097	0.0004
2006	Low	Free/Reduced	0.0096	0.0004
2006	High	Paid	0.0086	0.0003
2006	Low	Paid	0.0080	0.0003
2007	High	Free/Reduced	0.0106	0.0005
2007	Low	Free/Reduced	0.0110	0.0005
2007	High	Paid	0.0086	0.0003
2007	Low	Paid	0.0113	0.0003
2008	High	Free/Reduced	0.0094	0.0005
2008	Low	Free/Reduced	0.0079	0.0004
2008	High	Paid	0.0097	0.0003
2008	Low	Paid	0.0071	0.0003

Notes: The high specification initializes the probability at the minimum of 1 and the equilibrium for the previous deduction rules plus .2. The low specification initializes the probability at the maximum of 0 and the equilibrium for the previous deduction minus .2. The Max Difference column computes the maximum absolute difference between the equilibrium probability of admission for any school/ranked order list/priority group/deduction rule combination for the given initialization type and the standard initialization. The Mean Difference column computes the mean of the absolute value of differences for all school/ranked order list/priority group/deduction rule combinations.

APPENDIX B

APPENDIX B: SUPPLEMENTARY MATERIAL FOR BROAD VALIDITY OF THE FIRST-ORDER APPROACH IN MORAL HAZARD

B.1. Proofs

B.1.1. Solution of the Relaxed Problem

Throughout this subsection, fix a reservation utility $\bar{U}$ and regular intended action a_0. The relaxed problem only has a solution if $U(v, a)$ can be differentiated under the integral sign, so we begin by showing that differentiation under the integral at any action is possible for any feasible contract.

Remark B.1. *Let v be a feasible contract. Then, for all a in $\mathbb{R}_+$, $U_a(v, a)$ exists, is finite, and*

$$U_a(v, a) = \int v(y) f_a(y|a)\, dy.$$

Proof. The conditions for Leibniz's rule to hold are in assumption 2.2.2. □

We now derive some intermediate results to establish proposition 2.1. Recall the definition of the Lagrangian from equation (2.2) and definition 2.2 of a canonical contract.

Note that canonical contracts uniquely minimize the Lagrangian:

Lemma B.1. *Given λ and μ in $\mathbb{R}$, there exists v that minimizes the Lagrangian $\mathcal{L}(v, \lambda, \mu)$ among all feasible contracts. v is $f(\cdot|a_0)$ almost everywhere equal to the canonical contract $V(y, \lambda, \mu)$.*

Proof. The Lagrangian in equation (2.2) can be written as

$$\mathcal{L}(v, \lambda, \mu) = \int [k(v(y)) - \lambda v(y) - \mu v(y) S(y|a_0)] f(y|a_0)\, dy + \bar{U}.$$

Differentiating the integrand with respect to $v(y)$ yields

$$[k'(v(y)) - \lambda - \mu S(y|a_0)]f(y|a_0)$$

and this is strictly convex in $v(y)$. Therefore, the integrand is minimized pointwise in $v(y)$ at $v(y) = V(y|\lambda, \mu)$. Hence, the infimum is attained, and any minimizer satisfies the desired formula $f(y|a_0)$ almost everywhere.

It only remains to show that $V(y|\lambda, \mu)$ is feasible. This follows from assumption 2.2 Part 4. □

We now note that, given λ there is a unique value of μ that solves the local IC constraint.

Lemma B.2. *Given λ in $\mathbb{R}$, there exists a unique $\tilde{\mu}(\lambda)$ such that the canonical contract $V(y|\lambda, \tilde{\mu}(\lambda))$ satisfies the local IC constraint (LIC). Moreover, $\tilde{\mu}(\lambda) > 0$.*

Proof. We have

$$U_a(v) = \int v(y) f_a(y|a_0)\, dy.$$

If $\mu = 0$, then $v(y)$ is constant, so

$$U_a(V(\cdot|\lambda, 0, a_0)) - c'(a_0) < 0.$$

As $\mu \to \infty$, $V(y|\lambda, \mu, a_0)$ converges pointwise to $u(\infty)$ if $S(y|a_0) > 0$ and to $u(0)$ if $S(y|a_0) < 0$). Hence, because a_0 is regular, for large enough μ,

$$U_a(V(\cdot|\lambda, \mu, a_0)) - c'(a_0) > 0.$$

Therefore, there exists at least one $\mu_1 > 0$ such that

$$U_a(V(\cdot|\lambda, \mu_1, a_0)) - c'(a_0) = 0.$$

It only remains to prove that this solution μ_1 is unique. To see this, note that

$$U_a(V(\cdot|\lambda, \mu_1, a_0))$$

is weakly increasing in μ. And, moreover, it is strictly increasing at any solution μ_1 because $c'(a_0) > 0$ implies that there is a positive measure of y such that $f_a(y|a_0) > 0$ and

$$\lambda + \mu_1 \frac{f_a(y|a_0)}{f(y|a_0)} > \frac{1}{u'(0)}.$$

□

The lemma implies that the family of canonical contracts that satisfy (LIC) is a one-dimensional family indexed by λ. Define

$$\tilde{V}(y|\lambda) := V(y|\lambda, \tilde{\mu}(\lambda))$$

Define the **relaxed Pareto problem** as finding v in $\mathcal{C}$ to

$$\begin{aligned} \text{minimize} \quad & W(v) - \lambda U(v) \\ \text{subject to} \quad & \partial_a U(v, a_0) = 0 \end{aligned} \tag{LIC}$$

The next lemma shows that the contracts $\tilde{V}(y|\lambda)$ span the solutions to the Pareto problem:

Lemma B.3. *Given λ in $\mathbb{R}$, the relaxed Pareto problem has a solution, and any solution is $f(y|a_0)$ almost everywhere equal to $\tilde{V}(y|\lambda)$.*

Proof. For any v satisfying (LIC),

$$W(v) - \lambda U(v) = \mathcal{L}(v, \lambda, \tilde{\mu}(\lambda)) - \lambda \bar{U}.$$

Lemma B.1 then implies that $W(v) - \lambda(v)$ is minimized over $\mathcal{C}$ by $\tilde{V}(y|\lambda)$, and that this solution

is almost-everywhere unique. This contract satisfies (LIC) by the definition of $\tilde{\mu}(\lambda)$. □

Define the expected wage and utility attained by these contracts as

$$\begin{aligned}\tilde{U}(\lambda) &:= U(\tilde{V}(\cdot|\lambda), a_0),\\ \tilde{W}(\lambda) &:= W(\tilde{V}(\cdot|\lambda), a_0).\end{aligned}$$

Lemma B.4. *$\tilde{U}(\lambda)$ is strictly increasing.*

Proof. Consider $\lambda_1 < \lambda_2$ with optima $v_1 := \tilde{V}(\cdot|\lambda_1)$ and $v_2 := \tilde{V}(\cdot|\lambda_2)$. By optimality,

$$W(v_1) - \lambda_1 U(v_1) \leq W(v_2) - \lambda_1 U(v_2),$$

and

$$W(v_2) - \lambda_2 U(v_2) \leq W(v_1) - \lambda_2 U(v_1).$$

Adding the inequalities,

$$(\lambda_2 - \lambda_1) \cdot \Big(\tilde{U}(\lambda_2) - \tilde{U}(\lambda_1)\Big) \geq 0.$$

Therefore, $\tilde{U}$ is non-decreasing. It only remains to show that $\tilde{U}$ is strictly increasing. To reach a contradiction, assume that $\tilde{U}(\lambda_2) = \tilde{U}(\lambda_1)$. Optimality implies that $\tilde{W}(\lambda_1) = \tilde{W}(\lambda_2)$. Let $\bar{\lambda} = (\lambda_1 + \lambda_2)/2$. We have $\tilde{U}(\bar{\lambda}) = \tilde{U}(\lambda_1)$. By our assumptions on the score, v_1 and v_2 differ in a set of positive measure. By strict convexity of k, it follows that $\tilde{W}(\bar{\lambda}) < \tilde{W}(\lambda_1)$. This contradicts the optimality of v_1. □

The proof of proposition 2.1 follows from collecting these results.

Proof of Proposition 2.1. Let $\bar{U}_L := \tilde{U}(0)$ and $\bar{U}_R := \lim_{\lambda\to\infty} \tilde{U}(\lambda)$.

Part 1.

$\bar{U}_R$ equals $u(\infty) - c(a_0)$, so the relaxed problem is not feasible for $\bar{U} \geq \bar{U}_R$, as desired.

Part 2.

Let $\lambda^*(\bar{U})$ be 0 if $\bar{U} \leq \bar{U}_L$ and be the inverse of $\tilde{U}$ for $\bar{U}_L < \bar{U} < \bar{U}_R$. Lemma B.4 implies that λ^* is well defined and that $\lambda^*(\bar{U}) \geq 0$.

Let $\mu^*(\lambda) := \tilde{\mu}(\lambda^*(\bar{U}))$. Lemma B.2 implies that $\mu^*(\bar{U}) > 0$. Let $v^*(y|\bar{U}) := \tilde{V}(y|\lambda^*(\bar{U}))$. Note that this coincides with the definition of v^* in the proposition statement.

Lemma B.3 implies that $v^*(y|\bar{U})$ solves the Pareto problem given $\lambda^*(\bar{U})$. This implies that $v^*(y|\bar{U})$ also solves the relaxed cost minimization problem. Likewise, B.3 implies that the solution is unique almost everywhere.

Part 3.

This follows from the definition of $\lambda^*(\bar{U})$ which equals 0 in this range.

Part 4. λ^* strictly increasing follows from lemma $B.4$ and the strict convexity of ω follows from $v^*(\cdot|\lambda)$ being different for each value of λ and from k being strictly convex.

□

B.1.2. Proof of Proposition 2.2

This section demonstrates that the second derivative of agent's utility is negative for sufficiently high reservation utility. The bulk of the section is spent demonstrating the same result for λ sufficiently high, and the main result follows as an immediate corollary. Throughout this section, we use the $\tilde{V}(y|\lambda)$ notation because we are focused on the agent's problem as λ becomes large. Although μ is a function of λ (lemma B.2) we abuse notation by omitting μ's dependence on λ and using $\tilde{\mu}$ to denote $\tilde{\mu}(\lambda)$.

The section proceeds as follows. Lemmas B.5 and B.6 derive a convenient formula for the second derivative of agent's utility with respect to effort. Lemma B.7 shows that the agent's probability

of receiving payment approaches 1 as λ approaches infinity. Lemma B.8 shows that the second derivative of agent's utility is negative for λ sufficiently high. We then use these lemmas to prove the main result. Finally, we prove a few minor lemmas which we used in the proofs of the major lemmas.

We begin with an equation for the general form of agent's utility and its derivatives for an arbitrary contract.

Lemma B.5. *Given a contract, v, U and its derivatives evaluated at a equal*

$$\begin{aligned} U(v,a) &= \int v \cdot f(v|a)\, dy - c(a), \\ U_a(v,a) &= \int v \cdot S(y|a) f(y|a)\, dy - c'(a), \\ U_{aa}(v,a) &= \int v \cdot \left(S^2(y|a) + S_a(y|a)\right) f(v|a)\, dy - c''(a). \end{aligned}$$

Proof. The expression for $U(v,a)$ is just the definition. Differentiating $U(v,a)$ with respect to a yields

$$U_a(v,a) = \int v f_a(v|a)\, dv - c'(a).$$

The formula for U_a follows from the fact that $f_a(v|a) = f(v|a)S(v|a)$. Differentiating $U_a(v,a)$ with respect to a gives

$$\begin{aligned} U_{aa}(v,a) &= \int v \frac{\partial}{\partial a} \left[f(v|a)S(v|a)\right] dv - c'(a) \\ &= \int v \left[f_a(v|a)S(v|a) + f(v|a)S_a(v|a)\right] dv - c''(a) \\ &= \int v \left[f(v|a)S(v|a)S(v|a) + f(v|a)S_a(v|a)\right] dv - c''(a) \\ &= \int v \left[S^2(v|a) + S_a(v|a)\right] f(v|a)\, dv - c''(a). \end{aligned}$$

□

We now use lemma B.5 to derive the following equations for the agent's utility and its derivatives.

Lemma B.6. *Given a canonical contract, $v(y) := \tilde{V}(y|\lambda)$, the agent's utility and its derivatives evaluated at a are*

$$\begin{aligned}
U(v,a) &= g(\lambda) + \tilde{\mu}\int \Delta g(y|\lambda)\cdot S(y|a_0)f(y|a)\,dy - c(a),\\
U_a(v,a) &= \tilde{\mu}\int \Delta g(y|\lambda)\cdot S(y|a)\cdot S(y|a_0)f(y|a)\,dy - c'(a),\\
U_{aa}(v,a) &= \tilde{\mu}\int \Delta g(y|\lambda)\left(S^2(y|a) + S_a(y|a)\right)\cdot S(y|a_0)f(y|a)\,dy - c''(a),
\end{aligned}$$

where

$$\Delta g(y|\lambda) = \frac{g(\lambda + \tilde{\mu}S(y|a_0)) - g(\lambda)}{\tilde{\mu}S(y|a_0)}.$$

Proof. The utility function, U, evaluated at the canonical contract, $\tilde{V}(y|\lambda)$, and action, a, is

$$U\left(\tilde{V}(y|\lambda), a\right) = \int \tilde{V}(y|\lambda)f(y|a)\,dy.$$

Recall that $\tilde{V}(y|\lambda) = g\left(\lambda + \tilde{\mu}S(y|a_0)\right)$. Substituting in yields

$$U\left(\tilde{V}(y|\lambda), a\right) = \int g\left(\lambda + \tilde{\mu}S(y|a_0)\right) f(y|a)\,dy.$$

Adding and subtracting $g(\lambda)$, we rewrite the integral:

$$U\left(\tilde{V}(y|\lambda), a\right) = g(\lambda) + \int \left[g\left(\lambda + \tilde{\mu}S(y|a_0)\right) - g(\lambda)\right] f(y|a)\,dy.$$

Using the definition $\Delta g(y|\lambda) = \frac{g(\lambda+\tilde{\mu}S(y|a_0))-g(\lambda)}{\tilde{\mu}S(y|a_0)}$ and multiplying by $\frac{\tilde{\mu}S(y|a_0)}{\tilde{\mu}S(y|a_0)}$ yields

$$U\left(\tilde{V}(y|\lambda), a\right) = g(\lambda) + \tilde{\mu}\int \Delta g(y|\lambda)\cdot S(y|a_0)f(y|a)\,dy - c(a).$$

The lemma follows from B.5's formula for the derivatives applied to $v(y) = \Delta g(y|\lambda)\cdot S(y|a_0)$. □

We now demonstrate that for λ sufficiently large the agent receives a strictly positive payment for an arbitrarily large portion of the support of $f(y|a)$ for any $a > 0$. We begin by defining some

additional notation.

Definition B.1. *The **threshold score**, $\underline{S}(\lambda)$, is the maximum score such that the agent receives no payment. It is the score that solves*

$$\underline{S}(\lambda) = \frac{1}{\tilde{\mu}u'(0)} - \frac{\lambda}{\tilde{\mu}}.$$

The threshold score is well defined because $\tilde{\mu} > 0$ by lemma B.2.

Definition B.2. *The **threshold outcome**, $\underline{y}(\lambda)$, is the outcome which induces the threshold score, $\underline{S}(\lambda)$:*

$$\underline{y}(\lambda) = S^{-1}(\underline{S}(\lambda, \tilde{\mu})|a_0).$$

A score that satisfies the equation exists by assumption 2.3.3, which states that the score's image is $\mathbb{R}$.

Lemma B.7. *The threshold outcome approaches negative infinity as λ approaches infinity:*

$$\lim_{\lambda \to \infty} \underline{y}(\lambda) = -\infty.$$

Proof. Lemma B.6's result for U_a evaluated at a_0 yields

$$U_a\left(\tilde{V}(y|\lambda), a_0\right) = \tilde{\mu} \int \Delta g(y|\lambda) \cdot S(y|a_0)^2 f(y|a_0) - c'(a_0).$$

The local incentive compatibility constraint requires that $U_a\left(\tilde{V}(y|\lambda), a_0\right) = 0$. It follows that

$$\tilde{\mu} \int \Delta g(y|\lambda) \cdot S(y|a_0)^2 f(y|a_0) = c'(a_0).$$

Observe that the term inside the expectation is weakly positive because $\Delta g(y|\lambda) \geq 0$ by the monotonicity of g and $S(y|a_0)^2 \geq 0$. Therefore,

$$c'(a_0) \geq \tilde{\mu} \int_{S(y|a_0) \leq \underline{S}(\lambda)} \Delta g(y|\lambda) \cdot S^2(y|a_0) f(y|a_0)\, dy.$$

By definition, for all y in the domain of integration, $g(y) = u(0)$. Substituting in to $\Delta g(y|\lambda)$ yields

$$c'(a_0) \geq \tilde{\mu} \int_{S(y|a_0) \leq \underline{S}(\lambda)} \frac{(u(0) - g(\lambda))}{\tilde{\mu} S(y|a_0)} \cdot S^2(y|a_0) f(y|a_0)\, dy.$$

We simplify and use the fact that $f_a(y|a_0) = f(y|a_0)S(y|a_0)$

$$c'(a_0) \geq (u(0) - g(\lambda)) \cdot \int_{S(y|a_0) \leq \underline{S}(\lambda)} f_a(y|a_0)\, dy.$$

Integrating yields:

$$c'(a_0) \geq (u(0) - g(\lambda)) \cdot F_a(\underline{y}(\lambda)|a_0).$$

Observe that $u(0) - g(\lambda) \to -\infty$, and $F_a(\underline{y}(\lambda)|a_0) < 0$ by assumption 2.3.2. Suppose $F_a(\underline{y}(\lambda)|a_0)$ does not approach 0. Then we have $c'(a_0) \geq \infty$. By contradiction, $F_a(\underline{y}(\lambda)|a_0) \to 0$.

The threshold outcome, $\underline{y}(\lambda)$, cannot converge to infinity because $\underline{S}(\lambda)$ is negative for λ sufficiently large because $S(\infty|a_0) = \infty$ by assumption 2.3.3, and $\infty > \underline{S}(\lambda)$. If $\underline{S}(\lambda)$ does not converge to negative infinity, then there is a subsequence that converges to a finite number. However, $F_a(c|a_0) < 0$ for any c by assumption 2.3.2. The proposition is proven by contradiction. □

Lemma B.8. *As λ approaches infinity, the limit of the supremum of the second derivative of agent's utility at any action $a > 0$ is strictly negative:*

$$\limsup_{\lambda \to \infty} U_{aa}\left(\tilde{V}\left(y|\lambda\right), a\right) < 0.$$

Proof. By lemma B.6,

$$U_{aa}\left(\tilde{V}\left(y|\lambda\right), a\right) = g(\lambda) + \tilde{\mu} \int \Delta g(y|\lambda) \cdot S(y|a_0) f(y|a)\, dy - c(a)$$

Lemma B.9 states that there exists a y_0 such that the integrand is negative for all $y < y_0$. It follows

that

$$U_{aa}\left(\tilde{V}\left(y|\lambda\right),a\right) \leq \tilde{\mu}\int_{y_0}^{\infty} \Delta g(y|\lambda)\left(S^2(y|a)+S_a(y|a)\right)\cdot S(y|a_0)\cdot f(y|\hat{a})\,dy - c''(a).$$

Let Y_+ be the set of $y \geq y_0$ where this integrand is positive. Then

$$U_{aa}\left(\tilde{V}\left(y|\lambda\right),a\right) \leq \tilde{\mu}\int_{Y_+} \Delta g(y|\lambda)\left(S^2(y|a)+S_a(y|a)\right)\cdot S(y|a_0)\cdot f(y|\hat{a})\,dy - c''(a).$$

Lemma B.7 implies that for λ sufficiently large, $\underline{y}(\lambda, a_0) \leq y_0$. Recall that for $y > \underline{y}$, $g(y) = k'^{-1}(\lambda + \mu S(y|a_0))$. The concavity of k'^{-1} (assumption 2.4) implies that Δg is decreasing for y in Y_+. Therefore,

$$U_{aa}\left(\tilde{V}\left(y|\lambda\right),a\right) \leq \tilde{\mu}\Delta g(y_0|\lambda)\int_{Y_+} \left(S^2(y|a)+S_a(y|a)\right)\cdot S(y_0|a_0)\cdot f(y|\hat{a})\,dy - c''(a).$$

Because $g(y)$ is concave for $y > \underline{y}$ and $y_0 > \underline{y}$

$$\begin{aligned}\tilde{\mu}\Delta g(y_0|\lambda) &\leq \tilde{\mu} g'(\lambda + \tilde{\mu}S(y|a_0)) \\ &= \frac{\tilde{\mu}}{\lambda + \tilde{\mu}S(y|a_0)}\cdot(\lambda + \tilde{\mu}S(y|a_0))g'(\lambda + \tilde{\mu}S(y|a_0)).\end{aligned}$$

Lemma B.10 states that $\frac{\tilde{\mu}}{\lambda} \to 0$ as $\lambda \to \infty$, and assumption 2.4 implies that $(\lambda + \tilde{\mu}S(y_0|a_0))g'(\lambda + \tilde{\mu}S(y_0|a_0))$ has a finite limit. It follows that

$$U_{aa}\left(\tilde{V}\left(y|\lambda\right),a\right) \leq -c''(a) < 0$$

with the last inequality implied by the strict convexity of the cost function (assumption 2.1). □

We now prove proposition 2.2.

Proof. We first prove that there exists λ_0 such that, for all $\lambda \geq \lambda_0$ and a in $\mathcal{A}$,

$$U_{aa}(\tilde{V}(y|\lambda), a) \leq 0.$$

To reach a contradiction, assume that this is not the case. Then there exists a sequence of $\lambda_n \to \infty$ and a_k such that

$$U_{aa}(\tilde{V}(y|\lambda_n),_n) > 0.$$

Because $\mathcal{A}$ is compact, we can take a convergent subsequence where $a_k \to a_1$. Therefore,

$$\limsup_{\lambda \to \infty} U_{aa}(\tilde{V}(y|\lambda), a_1) > 0.$$

This contradicts lemma B.8.

For any $\lambda \geq \lambda_0$, we thus have that $U(\tilde{V}(y|\lambda), a)$ is concave in a. The proposition is proven by letting U^* be the solution to $\lambda^*(U^*) = \lambda_0$. The agent's problem is concave in a for $\bar{U} \in [U^*, U_R]$ because $\lambda^*(\bar{U})$ is monotonic by proposition 2.1.4 □

We conclude the section by proving two minor lemmas that we used in the above proofs.

Lemma B.9. *There exists y_0 such that for all $y < y_0$,*

$$\Delta g(y|\lambda) \left(S^2(y|a) + S_a(y|a)\right) \cdot S(Y|a_0) \leq 0.$$

Proof. The lemma follows from these 3 facts.

1. $\Delta g(y|\lambda) \geq 0$
2. There exists y_0 such that $S(Y|a_0) < 0$ for $y < y_0$
3. There exists y_0 such that $\left(S^2(y|a) + S_a(y|a)\right) > 0$ for $y \leq y_0$.

The monotonicity of g implies the first fact. The second fact follows from assumption 2.3.3. Assumption 2.3.4 and the following algebra implies the final fact.

$$S^2(y|\hat{a}) + S_a(y|a) = S^2(y|\hat{a}) + \frac{d}{da}\frac{f_a(y|a)}{f(y|a)} = S^2(y|\hat{a}) + \frac{f_{aa}(y|a)}{f(y|a)} - \frac{f_a(y|a)^2}{f(y|a)^2} = \frac{f_{aa}(y|a).}{f(y|a)}.$$

The density is always positive, so the final expression is positive whenever $f_{aa}(y|a)$ is positive. □

Lemma B.10. *As* $\lambda \to \infty$, $\frac{\tilde{\mu}}{\lambda} \to 0$.

Proof. Lemma B.7 and assumption 2.3.3 imply that $\underline{S}(\lambda) \to -\infty$. The result follows from the definition of $\underline{S}(\lambda)$. □

B.1.3. Proof of Theorem 2.1

Parts (1) and (2) follow from proposition 2.1. Part (3) follows from proposition 2.2. For any $\bar{U} \geq U^*$ we have that $v^*(\bar{U})$ is concave, and thus is a solution to the cost minimization problem. Any other solution also solves the relaxed cost minimization problem, so equals $v^*(\bar{U})$ almost everywhere.

Functional Form Derivations

B.2. Distributions

B.2.1. Normal

The PDF is

$$f(y|a,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right).$$

The derivative with respect to a is

$$f_a(y|a,\sigma^2) = f(y|a,\sigma^2)\frac{y-a}{\sigma^2}.$$

The score is

$$S(y|a) = \frac{f(y|a,\sigma^2)\frac{y-a}{\sigma^2}}{f(y|a,\sigma^2)} = \frac{y-a}{\sigma^2}.$$

B.2.2. Log Normal

The PDF is

$$f(y|a,\sigma^2) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(y)-a)^2}{2\sigma^2}\right).$$

The derivative with respect to a is

$$f_a(y|a,\sigma^2) = f(y|a,\sigma^2)\frac{log(y)-a}{\sigma^2}.$$

The score is

$$S(y|a) = \frac{f(y|a,\sigma^2)\frac{log(y)-a}{\sigma^2}}{f(y|a,\sigma^2)} = \frac{log(y)-a}{\sigma^2}.$$

B.2.3. Poisson

The PMF is

$$P(y|a) = \frac{a^y e^{-a}}{y!}.$$

The derivative with respect to a is

$$P_a(y|a) = \frac{ya^{y-1}e^{-a}}{y!} - \frac{a^y e^{-a}}{y!}.$$

Simplifying yields

$$P_a(y|a) = P(y|a)(\frac{y}{a} - 1) = f(a)\frac{y-a}{a}.$$

The score is

$$S(y|a) = \frac{f(a)\frac{y-a}{a}}{f(a)} = \frac{y-a}{a}.$$

B.2.4. Exponential

The PDF is

$$f(y|a) = \frac{1}{a}e^{-\frac{y}{a}}.$$

Observe that $\frac{\partial a}{\partial}\frac{1}{a} = -\frac{1}{a^2}$, and $\frac{\partial a}{\partial}e^{-\frac{y}{a}} = \frac{y}{a^2}e^{-\frac{y}{a}}$. Using the product rule,

$$f_a(y|a) = \frac{y}{a^3}e^{-\frac{y}{a}} - \frac{1}{a^2}e^{-\frac{y}{a}}.$$

Simplifying yields

$$f_a(y|a) = \frac{y-a}{a^2}f(y|a).$$

The score is

$$S(y|a) = \frac{y-a}{a^2}.$$

B.2.5. Bernoulli

The PMF is

$$P(y|a) = a^y(1-a)^{1-y}, \text{ where } y \in \{0,1\}.$$

The derivative with respect to a is

$$P_a(y|a) = ya^{y-1}(1-a)^{1-y} - (1-y)a^y(1-a)^{-y}.$$

Simplifying yields

$$P_a(y|a) = P(y|a)(\frac{y}{a} - \frac{1-y}{1-a}) = P(y|a)\frac{y-a}{a-a^2}.$$

The score is

$$S(y|a) = \frac{P(y|a)\frac{y-a}{a-a^2}}{P(y|a)} = \frac{y-a}{a-a^2}.$$

B.2.6. Geometric

The PMF is

$$P(y|a) = (1-\frac{1}{a})^{y-1}(\frac{1}{a}).$$

The derivative with respect to a is

$$P_a(y|a) = (y-1)(\frac{a-1}{a})^{y-2}(\frac{1}{a^3}) - (1-\frac{1}{a})^{y-1}(\frac{1}{a^2}).$$

Simplifying yields

$$P_a(y|a) = \frac{(y-1)a}{a-1}(\frac{a-1}{a})^{y-1}\frac{1}{a^3} - P(y|a)\frac{1}{a}.$$

We simplify further

$$P_a(y|a) = \frac{(y-1)}{(a-1)a}P(y|a) - P(y|a)\frac{1}{a}.$$

We factor out $P(y|a)$

$$P_a(y|a) = P(y|a)\frac{(y-a)}{(a^2-a)}.$$

The score is

$$S(y|a) = \frac{P(y|a)\frac{(y-a)}{(a^2-a)}}{P(y|a)} = \frac{y-a}{a^2-a}$$

B.2.7. Binomial

The PMF is

$$P(y|a,n) = \binom{n}{y}a^y(1-a)^{n-y}.$$

The derivative with respect to a is

$$P_a(y|a) = y\binom{n}{y}a^{y-1}(1-a)^{n-y} - (n-y)\binom{n}{y}a^y(1-a)^{n-y-1}.$$

Simplifying yields

$$P_a(y|a) = \frac{y}{a}P(y|a,n) - \frac{n-y}{1-a}P(y|a,n).$$

I factor out $P(y|a)$ and simplify further

$$P_a(y|a) = P(y|a,n)\frac{y-na}{a-a^2}.$$

The score is

$$S(y|a) = \frac{P(y|a)\frac{(y-na)}{(a-a^2)}}{P(y|a)} = \frac{y-na}{a-a^2}.$$

B.2.8. Gamma

The PDF is

$$f(y|n,a) = \frac{y^{n-1}e^{-y/a}}{\Gamma(n)a^n}.$$

The derivative with respect to a is

$$f_a(y|n,a) = \frac{y}{a^2}f(y|n,a) - \frac{n}{a}f(y|n,a).$$

I factor out $f(y|n,a)$ and simplify

$$f_a(y|n,a) = f(y|n,a)\frac{y-na}{a^2}.$$

The score is

$$S(y|a) = \frac{f(y|n,a)\frac{y-na}{a^2}}{f(y|n,a)} = \frac{y-na}{a^2}.$$

B.2.9. t

The PDF is

$$f(y|\nu,a,\sigma) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

The derivative with respect to a is

$$f_a(y|\nu,a,\sigma) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\frac{\nu+1}{2}\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)^{-\frac{\nu+3}{2}}\frac{2(y-a)}{\nu\sigma^2}.$$

Simplifying yields

$$f_a(y|\nu,a,\sigma) = f(y|\nu,a,\sigma)\frac{(\nu+1)(y-a)}{\nu\sigma^2\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)} = f(y|\nu,a,\sigma)\frac{(\nu+1)(y-a)}{\nu\sigma^2+(y-a)^2}.$$

The score is

$$S(y|a) = \frac{f(y|n,a)\frac{(\nu+1)(y-a)}{\nu\sigma^2+(y-a)^2}}{f(y|n,a)} = \frac{(\nu+1)(y-a)}{\nu\sigma^2+(y-a)^2}.$$

B.2.10. Exponential Family

The PDF is

$$f(y|a) = h(y)\exp\left(\eta(a)T(y) - A(a)\right),$$

where $h(y)$ is the base measure, $\eta(a)$ is the natural parameter, $T(y)$ is the sufficient statistic, and $A(a)$ is the log-partition function. The derivative with respect to a is

$$f_a(y|a) = h(y)\exp\left(\eta(a)T(y) - A(a)\right)\left[T(y)\frac{d\eta(a)}{da} - \frac{dA(a)}{da}\right].$$

Simplifying yields

$$f_a(y|a) = f(y|a)\left[T(y)\frac{d\eta(a)}{da} - \frac{dA(a)}{da}\right].$$

The score is

$$S(y|a) = \frac{f_a(y|a)}{f(y|a)} = T(y)\frac{d\eta(a)}{da} - \frac{dA(a)}{da}.$$

B.2.11. Additive

Let $y = a + y$, where y is a random variable with PDF $g(y)$. The PDF of y is

$$f(y|a) = g(y - a).$$

The derivative with respect to a is

$$f_a(y|a) = -g'(y - a).$$

The score is

$$S(y|a) = \frac{g'(y - a)}{g(y - a)}.$$

B.2.12. Multiplicative

Let $y = ay$ where y is a random variable with PDF $g(y)$. The PDF of y is

$$f(y|a) = \frac{1}{a}g(\frac{y}{a}).$$

The derivative with respect to a is

$$-\frac{1}{a^2}g(\frac{y}{a}) - \frac{1}{a^3}g'(\frac{y}{a})y.$$

The score is

$$S(y|a) = -\frac{1}{a} - \frac{1}{a^2}\frac{g'(\frac{y}{a})}{g(\frac{y}{a})}y$$

B.3. Utility Functions

B.3.1. Log

This section derives results for the utility function

$$u(x) = log(w_0 + x).$$

The derivative of utility with respect to consumption is

$$u'(x) = \frac{1}{w_0 + x}$$

The cost of utility is the inverse of the utility function

$$u = log(w_0 + x) \rightarrow exp(u) = x \rightarrow k(u) = exp(u) - w_0.$$

The marginal cost of utility is

$$\frac{1}{u'(k(u))} = \frac{1}{\frac{1}{exp(u)}} = exp(u).$$

The inverse of the marginal cost of utility is

$$x = exp(u) \rightarrow k'^{-1}(x) = log(x).$$

The link function, $g(z)$, is

$$g(z) = k'^{-1}\left(\max\left(\frac{1}{u'(w_0)}, z\right)\right) = \log\left(\max(w_0, z)\right).$$

The canonical wage function given $z = \lambda + \mu S(y|a_0)$ is

$$\begin{aligned} w(z) &= k(g(z)) \\ &= \exp\big(\log(max(w_0, z))\big) - w_0 \\ &= (z - w_0)^+ . \end{aligned}$$

B.3.2. CRRA

This section derives results for the utility function

$$u(x) = \frac{(w_0 + x)^{1-\gamma}}{1-\gamma}.$$

The derivative of utility with respect to consumption is

$$u'(x) = (w_0 + x)^{-\gamma}.$$

The cost of utility is the inverse of the utility function

$$u = \frac{(w_0 + x)^{1-\gamma}}{1-\gamma} \rightarrow k(u) = ((1-\gamma)u)^{\frac{1}{1-\gamma}} - w_0.$$

The marginal cost of utility is

$$\frac{1}{u'(k(u))} = \frac{1}{((1-\gamma)u)^{\frac{-\gamma}{1-\gamma}}} = ((1-\gamma)u)^{\frac{\gamma}{1-\gamma}}.$$

The inverse of the marginal cost of utility is

$$x = ((1-\gamma)u)^{\frac{\gamma}{1-\gamma}} \rightarrow k'^{-1}(x) = \frac{x^{\frac{1-\gamma}{\gamma}}}{1-\gamma}.$$

The link function, $g(z)$, is

$$g(z) = k'^{-1}\left(\max\left(\frac{1}{u'(w_0)}, z\right)\right) = \frac{\max\left(w_0^{\gamma}, z\right)^{\frac{1-\gamma}{\gamma}}}{1-\gamma}.$$

The canonical wage function given $z = \lambda + \mu S(y|a_0)$ is

$$\begin{aligned} w(z) &= k(g(z)) \\ &= \left(\frac{1-\gamma}{1-\gamma} \cdot \max(w_0^\gamma, z)^{\frac{1-\gamma}{1-\gamma}} \right)^{\frac{1}{1-\gamma}} - w_0 \\ &= max(w_0^\gamma, z)^{\frac{1}{\gamma}} - w_0 \\ &= \left((z^+)^{\frac{1}{\gamma}} - w_0 \right)^+. \end{aligned}$$

B.3.3. CARA

This section derives results for the utility function

$$u(x) = \frac{-\exp(-\alpha(x + w_0))}{\alpha}.$$

The derivative of utility with respect to consumption is

$$u'(x) = exp(-\alpha(x + w_0)).$$

The cost of utility is the inverse of the utility function

$$u = \frac{-\exp(-\alpha(x + w_0))}{\alpha} \rightarrow -\alpha u = exp(-\alpha(x + w_0)) \rightarrow k(u) = -\frac{log(-\alpha u)}{\alpha} - w_0.$$

The marginal cost of utility is

$$\frac{1}{u'(k(u))} = \frac{1}{exp(\alpha \frac{log(-\alpha u)}{\alpha})} \rightarrow k'(u) = -\frac{1}{\alpha u}.$$

The inverse of the marginal cost of utility is

$$x = -\frac{1}{\alpha u} \rightarrow k'^{-1}(x) = -\frac{1}{\alpha x}.$$

The link function, $g(z)$, is

$$g(z) = k'^{-1}\left(\max\left(\frac{1}{u'(w_0)}, z\right)\right) = -\frac{1}{\alpha \max\left(\exp(\alpha w_0), z\right)}.$$

The canonical wage function given $z = \lambda + \mu S(y|a_0)$ is

$$\begin{aligned}
w(z) &= k(g(z)) \\
&= k\left(-\frac{1}{\alpha \max\left(\exp(\alpha w_0), z\right)}\right) \\
&= -\frac{\log\left(-\alpha \cdot \left(-\frac{1}{\alpha \max(\exp(\alpha w_0), z)}\right)\right)}{\alpha} - w_0 \\
&= -\frac{(log(1) - \log\left(\max\left(\exp(\alpha w_0), z\right)\right))}{\alpha} - w_0 \\
&= \frac{\log\left(\max\left(\exp(\alpha w_0), z\right)\right)}{\alpha} - w_0 \\
&= \frac{(\log^+ z - \alpha w_0)^+}{\alpha}.
\end{aligned}$$

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